On the Connections between Radiation Synthesis and Sound Field Synthesis using Linear Arrays

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Introduction

Linear arrays (LA) are widely used for different sound reinforcement applications since the advent of electro-acoustics. The theory of radiation phenomena is well evolved. Nowadays LAs are used for full audio bandwidth public address of very large audiences, for speech bandwidth public address in highly reverberant environments, home entertainment and teleconferencing applications, as well as acoustic field holography such as sound field synthesis. With improved digital signal processing capabilities all approaches aim at full electronic control of the LA's individual sources. We provide a short review of the fundamentals of LA radiation recalling the obvious connections between sound field synthesis (SFS) and radiation synthesis (RS).

Linear Array Radiation

Consider a LA located on the y-axis $\mathbf{x}_0 = (0, y_0, 0)^T$ and a listening region $\mathbf{x} = (x > 0, y, z)^T$. The spatio-temporal spectrum $P(x, k_y, z, \omega)$ of the sound pressure function w.r.t. space and time $p(\mathbf{x}, t)$ is given by the Fourier transform pair, e.g. [1, (44,45)]

$$P(x,k_y,z,\omega) = \iint\limits_{-\infty}^{+\infty} p(\mathbf{x},t)\,\mathrm{e}^{+\mathrm{j}\,k_y\,y}\,\mathrm{d}y\,\mathrm{e}^{-\mathrm{j}\,\omega\,t}\,\mathrm{d}t, \tag{1}$$

$$p(\mathbf{x},t) = \frac{1}{4\,\pi^2}\iint\limits_{-\infty}^{+\infty} P(x,k_y,z,\omega)\,\mathrm{e}^{-\mathrm{j}\,k_y\,y}\,\mathrm{d}k_y\,\mathrm{e}^{+\mathrm{j}\,\omega\,t}\,\mathrm{d}\omega.$$

Following the signal processing model in Fig. 1, different spatio-temporal sound pressure spectra of the sound field generated by LAs can be obtained by different driving function's spatio-temporal spectra

$$\begin{Bmatrix} P(k_y,\omega) \\ P_w(k_y,\omega) \\ P_{w,\mathrm{S}}(k_y,\omega) \\ P_{w,\mathrm{S},H}(k_y,\omega) \end{Bmatrix} = \begin{Bmatrix} D(k_y,\omega) \\ D_w(k_y,\omega) \\ D_{w,\mathrm{S}}(k_y,\omega) \\ D_{w,\mathrm{S},H}(k_y,\omega) \end{Bmatrix} \cdot G_0(x,k_y,z,\omega). \tag{2}$$

$D(k_y, \omega)$ models an infinite, continuous LA; $D_w(k_y, \omega)$ a finite length, continuous LA; $D_{w,\mathrm{S}}(k_y, \omega)$ a finite length, equidistantly discretized LA built from spherical monopoles and $D_{w,\mathrm{S,H}}(k_y, \omega)$ a finite length, discretized LA built from identical non-isotropic sources. The latter follows from the well known product or pattern multiplication theorem of array processing [3, p.174], [4, Ch. 2.8]. Note that a LA built from spherical monopoles radiates an axial-symmetric sound field w.r.t. the y-axis. $G_0(x, k_y, z, \omega)$ is the spatio-temporal spectrum [1, (52)]

$$G_0(x,k_y,z,\omega) = \begin{cases} -\frac{\mathrm{j}}{4}\,H_0^{(2)}\left(\sqrt{\left(\frac{\omega}{c}\right)^2 - k_y^2}\cdot\sqrt{x^2+z^2}\right) & \text{for } k_y^2 < (\frac{\omega}{c})^2 \\ \frac{1}{2\,\pi}\,K_0\left(\sqrt{k_y^2 - \left(\frac{\omega}{c}\right)^2}\cdot\sqrt{x^2+z^2}\right) & \text{for } k_y^2 > (\frac{\omega}{c})^2, \end{cases} \tag{3}$$

of the Helmholtz equation's 3D free-field Green's function

$$G(\mathbf{x},\mathbf{x}_0,\omega) = \frac{1}{4\,\pi}\,\frac{\mathrm{e}^{-\mathrm{j}\,\frac{\omega}{c}\,\|\mathbf{x}-\mathbf{x}_0\|}}{\|\mathbf{x}-\mathbf{x}_0\|} \tag{4}$$

originating from $\mathbf{x}_0 = \mathbf{0}$ [1, (52)]. $H_0^{(2)}(\cdot)$ denotes the 0^{th} order Hankel function of 2^{nd} kind, $K_0(\cdot)$ the modified Bessel function of 0^{th} order of 2^{nd} kind [5, §10.1] and c the speed of sound. The 1^{st} case in (3) describes propagating waves, the 2^{nd} case corresponds to evanescent waves.

The propagating part of $G_0(x, k_y, z, \omega)$ is bounded and thus bandlimited to the region where $|k_y| < |\frac{\omega}{c}|$ allows propagating wave radiation. This is referred to as the *visible region* [4, Ch. 2.3] of the LA. Evanescent wave radiation occurs for $|k_y| > |\frac{\omega}{c}|$, this part of the spectrum is not bandlimited, however it is decaying rapidly for increased $\sqrt{x^2+z^2}$ and/or ω. For a discretized LA the spectra $D_{w,\mathrm{S}}(k_y, \omega)$ and $D_{w,\mathrm{S,H}}(k_y, \omega)$ include propagating spatial aliasing for $|k_y| < |\frac{\omega}{c}|$ from the propagating and evanescent spectral repetitions of the baseband of $D(k_y, \omega)$ or $D_w(k_y, \omega)$. For $|k_y| > |\frac{\omega}{c}|$ evanescent spatial aliasing is included from those contributions [6]. Note that the reconstruction/postfilter filter in the sampling model in Fig. 1 acts in the acoustic domain. Hence, loudspeakers with appropriate spatial lowpass characteristics and a spatial dimension not larger than the discretization step Δy may avoid or reduce propagating spatial aliasing [7,8].

By restricting the spatio-temporal driving function's spectra to the visible region $-|\frac{\omega}{c}| < k_y < +|\frac{\omega}{c}|$, the mapping between k_y and the propagating radiation angle φ

$$k_y = \frac{\omega}{c}\,\sin\varphi \tag{5}$$

allows for the interpretation of an angular spectrum synthesis, i.e. the superposition of 'plane' waves with $-\pi/2 < \varphi < +\pi/2$. For finite length LAs this leads to the frequency dependent farfield radiation patterns $D_w(\varphi, \omega)$, $D_{w,\mathrm{S}}(\varphi, \omega)$ and $D_{w,\mathrm{S,H}}(\varphi, \omega)$. $D_{w,\mathrm{S}}(\varphi)$ is usually referred to as the *array factor* [4, p.45], whereas

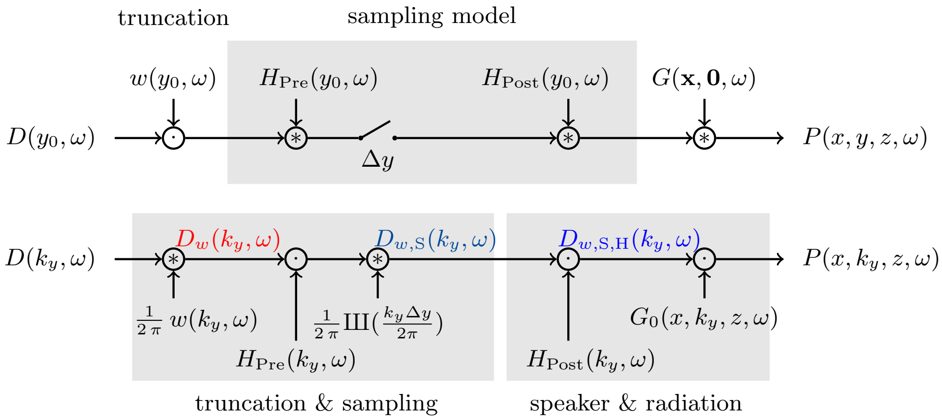

Figure 1: SFS & RS signal processing model in temporal (top) and spatio-temporal spectrum domain (bottom). Convolution w.r.t. y, k_y is denoted by $\circledast$ (not to be confused with the circular convolution), multiplication w.r.t. y, k_y by $\odot$, cf. [2, Fig. 5.13].

$D_{w,\mathrm{S,H}}(\varphi)$ is termed *final array factor*. The farfield radiation pattern completely describes the sound field

$$P(r,\varphi,\omega) \propto D(\varphi,\omega)\,\frac{\mathrm{e}^{-\mathrm{j}\,\frac{\omega}{c}\,r}}{r} \tag{6}$$

for distances $r = |\mathbf{x}|$ located in the Fraunhofer region, commonly termed farfield of the LA. The extent of the Fresnel region (nearfield) and transition towards the Fraunhofer region is highly dependent on the LA length, the frequency, the truncation window, the spatial discretization and the pre-/postfilter as well as – this important fact becomes occasionally ignored – the 'virtual' source to be synthesized. These parameters are all controlled by the complex driving function's spatio-temporal spectrum $D(k_y,\omega)$ respectively the spectrum $D_{w,\mathrm{S},H}(k_y,\omega)$ that finally determines the radiated sound field based on the chosen LA setup. Propagating spatial aliasing, for instance results in a severely corrupted Fresnel region. The sound field in this region is not amenable for equalization, since the sound pressure is highly dependent on the listener position and the frequency. Furthermore this aliasing contributions are included as grating lobes in the farfield radiation pattern [4, 9].

It is obvious that SFS and RS follow the same acoustic fundamentals by only choosing appropriate driving functions and LA configurations for the intended application. SFS applications are meaningful in the Fresnel region, whereas RS is typically approached in the Fraunhofer region only. A general and strict separation is not advisable. However, all approaches that consider broad audio bandwidth should aim at a frequency independent transition of the Fresnel/Fraunhofer region.

Sound Field Synthesis

SFS is typically considered for a horizontal target half plane $\mathbf{x} = (x > 0, y, z = 0)^T$. For linear LAs, implicit and explicit analytic solutions for driving functions were derived considering an infinite and continuous LA.

The implicit solution is given from the forward wave field propagator, namely the Rayleigh integral under Neumann boundary condition on the Green's function. This constitutes one possible solution of what is known as 2.5D Wave Field Synthesis (WFS). It can be derived from the 2D Rayleigh integral under a large argument approximation of the 2D Neumann Green's function. The driving function is known as, cf. e.g. [10, Ch. 2.4], [11, 12]

$$D(\mathbf{x}_{\mathrm{ref}},y_0,\omega) = -2\,\frac{\partial P(\mathbf{x},\omega)}{\partial x}\bigg|_{x=x_0}\sqrt{\frac{2\,\pi\,|\mathbf{x}_{\mathrm{ref}}-\mathbf{x}_0|}{\mathrm{j}\,\frac{\omega}{c}}}. \tag{7}$$

2.5D WFS requires a reference point or region for which the driving function synthesizes the desired sound field correctly. Either referencing to a single point $\mathbf{x}_{\mathrm{ref}} = (x_{\mathrm{ref}}, 0, 0)^T$ [12] or along a line parallel to the LA $\mathbf{x}_{\mathrm{ref}} = (x_{\mathrm{ref}}, y_0, 0)^T$ [1, 13] was discussed in literature.

The explicit solution – introduced as Spectral Division Method (SDM) [1] – is based on the inverse wave field propagator. It derives the solution from a desired sound field given along a reference line $x_{\mathrm{ref}} > x_0$ usually in the listening plane $z = 0$. Within the spatio-temporal spectral domain the deconvolution reads

$$D(x_{\mathrm{ref}},k_y,z,\omega) = \frac{P(x_{\mathrm{ref}},k_y,z=0,\omega)}{G_0(x_{\mathrm{ref}},k_y,z=0,\omega)}. \tag{8}$$

This is equivalent to Fourier Transform-Based Near-Field Acoustical Holography (Fourier-NAH) [14, Ch.3], [15, Ch. 6], [16] here applied for the 2.5D case. In [17, Ch. 3] the connection to a continuous singular value decomposition (SVD) problem was furthermore given. Note that even for a simple virtual point source the inverse spatial Fourier transform is not amenable for an exact analytic solution. Therefore, in literature different approximations were introduced to derive analytic driving functions for point sources [13]. WFS and SDM typically derive driving functions for infinite and continuous LAs. Spatial truncation and/or spatial sampling of the driving function is subsequently discussed separately [6], from which the farfield radiation pattern can be estimated [9].

Another possible solution for SFS considers finite length and discretized LAs directly. It numerically solves the equation system [18]

$$\mathbf{p} = \mathbf{G}\,\mathbf{d} + \mathbf{e}. \tag{9}$$

Very often this is performed for monochromatic wave radiation, where the $(M \times 1)$ vector $\mathbf{p}$ exhibits the complex sound pressure values of the desired sound field at M control points in front of the LA. The $(N \times 1)$ vector $\mathbf{d}$ consists of complex driving function weights $D[\mathbf{x}_{0,n}, \omega]$ from an N-sources LA and $\mathbf{G}$ is the $(M \times N)$ matrix including the acoustic transfer function (ATF) from the N sources $\mathbf{x}_0$ to the M control points $\mathbf{x}$ according to (4) in the free-field case using spherical monopoles. Note that the ATF can also include real loudspeaker and room characteristics. Different solutions are known for $\mathbf{d}$ under minimization of the error $\mathbf{e} = \mathbf{p} - \mathbf{G}\,\mathbf{d}$ using cost functions involving different norms and constraints appropriate to the intended application, cf. e.g. [19–24]. Since typically $M > N$ the inverse problem is ill-posed and appropriate regularization techniques such as a truncated or weighted SVD (WSVD, TSVD) or standard Tikhonov regularization (TR) [16] must be employed to provide stable results. Especially, for numerical SFS it is advisable to check the resulting farfield radiation pattern from the obtained driving function for tolerated side and grating lobe contributions.

Radiation Synthesis

RS became an early adopted treatment since the advent of electro-acoustics [3, 25, 26]. It aims at an appropriate design of a desired farfield radiation pattern $D_{w,\mathrm{S,H}}(\varphi, \omega)$, i.e. the final array factor. Thus the optimization of the Fraunhofer region is of interest. Typically a finite length LA using spherical monopoles serves as the staring point in array processing and antenna design. For a LA with N discrete source positions $\mathbf{x}_{0,n}$ the spatio-temporal spectrum is well known as a DTFT [27]

$$D_{w,\mathrm{S}}(k_y, \omega) = \sum_{n}^{n+N-1} D[\mathbf{x}_{0,n}, \omega]\, \mathrm{e}^{+\mathrm{j}\,\Omega\, n} \qquad (10)$$

of the complex weights $D[\mathbf{x}_{0,n}, \omega]$ for a single frequency ω. For the visible region $-\frac{\pi}{2} < \varphi < +\frac{\pi}{2}$, i.e. the angular frequency range $(-|\frac{\omega}{c}|\,\Delta y) < \Omega < (+|\frac{\omega}{c}|\,\Delta y)$ of the DTFT, the farfield radiation pattern is derived as

$$D_{w,\mathrm{S}}(\varphi, \omega) = D_{w,\mathrm{S}}(k_y, \omega)\big|_{\Omega = \frac{\omega}{c}\,\sin\varphi\,\Delta y}. \qquad (11)$$

The relation $\frac{\Delta y}{\lambda}$ determines which part of the DTFT spectrum on the unit circle corresponds to the visible region. For $\lambda < \frac{\Delta y}{2}$, i.e. fulfilling the spatial sampling theorem only a part of DTFT's base band is considered. For $\lambda = \frac{\Delta y}{2}$ the whole unit circle maps into the visible region. For $\lambda > \frac{\Delta y}{2}$ the unit circle is repeatedly used resulting in spectral repetitions of the DTFT baseband that contribute as propagating spatial aliasing [28].

Due to the relationship of a finite length sequence of driving weights and its DTFT spectrum all well known FIR filter design methods [29] may be employed to derive appropriate farfield radiation patterns, either in the sequence's or its DTFT domain ($\lambda < \frac{\Delta y}{2}$ must hold). Consequently this aims at methods of placing zeros within the z-plane. One of the most simple RS algorithms is the Delay-and-Sum beam former [4, Ch. 2.5] – here given for a LA symmetric to the origin –

$$D_{w=rect,\mathrm{S}}(k_y, \omega) = \frac{\sin([k_y - k_{y,\mathrm{Steer}}]\,\Delta y\, N/2)}{\sin([k_y - k_{y,\mathrm{Steer}}]\,\Delta y/2)} \qquad (12)$$

using (5) for the steering angle φ_{Steer}, i.e. $k_{y,\mathrm{Steer}} = \frac{\omega}{c}\sin\varphi_{\mathrm{Steer}}$. This technique is often combined with frequency dependent, parametric windowing in order to control the main lobe beam width vs. side lobe gain trade-off. For RS, windows such as the Dolph-Chebychev, Taylor or based on discrete prolate spheroidal sequences (DPSS) [30] – which can be approximated by the Kaiser-Bessel window – have been widely used.

Undirected RS is another well researched field. By realizing that the power-spectral-density (PSD) $|D_{w,\mathrm{S}}(k_y, \omega)|^2$ corresponds to the autocorrelation function (ACF) of the – here – frequency independent driving weights, sequences are looked for, that exhibit an Dirac-like ACF. This consequently results in a constant PSD and therefore in a frequency independent farfield radiation pattern. The method became known as allpass arrays [27], for which Barker codes [31] and other (pseudo)-binary sequences [32] were utilized for loudspeaker arrays. The so called Bessel array [33] is a special case of an allpass array [27]. Besides that, numerical optimized RS is still an ongoing research topic, cf. e.g. [34, 35].

Note that in contrast to antenna design, RS in acoustics is much more challenging. For full audio bandwidth, the wave lengths are typically much larger and simultaneously much smaller than common array dimensions. Therefore, providing frequency-independent beam patterns and thus the same Fresnel/Fraunhofer transition is a demanding task for large scale sound reinforcement.

Conclusion

This paper provided a short overview on the concepts of sound field and radiation synthesis. It was re-emphasized, that the same multi-dimensional acoustic signal processing model holds. The radiation phenomena can be conveniently discussed within the spatio-temporal and angular spectrum domain. The desired application determines the appropriate driving function and array setup, which for the Fresnel and Fraunhofer regions must be examined in detail.

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