A TUTORIAL ON THE REPRESENTATION OF TWO-DIMENSIONAL WAVE FIELDS BY MULTIDIMENSIONAL SIGNALS

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ABSTRACT

The representation of information by waves is common to many fields in engineering and physics like electromagnetism, acoustics, and optics. Many different tools for the description of wave-like signals have been developed. Most of them are based on methods and results from mathematical physics with different viewpoints according to the specific application areas. This tutorial presents a framework for some of the most common of these representations. It is based on well-known transform domain signal descriptions from multidimensional systems theory.

1. INTRODUCTION

Waves play a dominant role in the transmission of information over time and space. Their propagation is governed by the wave equation. It is the foundation of many applications in acoustics, optics, and electromagnetism. Solutions of the wave equation are called wave fields.

This tutorial derives various representations of wave fields. The purpose is to provide a framework for the essential relations with only a basic knowledge in signals and systems. The components of this framwork are the wave equation and Fourier transforms, Fourier series, and Dirac impulses in one and two dimensions. This contribution is an abridged version of [1].

The representations in this tutorial are embedded in a time-dependent, spatially three-dimensional description. However, only an important special case is considered here, namely that the spatial signals are separable in one of the three spatial coordinates. This assumption allows a reduction to two-dimensional wave fields which is important for all applications where the transmitters and receivers are located in a plane. Therefore, only the two-dimensional case is presented here.

Sections 2 and 3 review some well-known facts about one-dimensional signals and systems and introduce the same notions for multidimensional signals and systems, respectively. Sec. 4 specifies these results for wave fields, i.e. for signals which satisfy the wave equation. The so-called plane wave solution is considered in detail and it is shown how to represent also the general solution of the wave equation by plane waves. The resulting representations are derived for different coordinate systems. Finally the results are compiled to highlight the developed framework for the representation of two-dimensional wave fields.

2. ONE-DIMENSIONAL SIGNALS

This section reviews some well-known one-dimensional signal transforms with respect to time, space, angular, and radial coordinates. These remarks serve as reference for the multidimensional case in Sec. 3.

2.1. Fourier Transform

For the time variable t and the temporal frequency variable ω, the *Fourier transform* is given by

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \;, \qquad (1)$$

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega \;. \qquad (2)$$

The Fourier transform applies also to functions of the space variable x and the spatial frequency variable k_x

$$\mathcal{T}\{f(x)\} = \tilde{f}(k_x) = \int_{-\infty}^{\infty} f(x) e^{-jk_x x}\, dx \;, \qquad (3)$$

$$\mathcal{T}^{-1}\{\tilde{f}(k_x)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k_x) e^{jk_x x}\, dk_x \;. \qquad (4)$$

2.2. Fourier Series

Periodic functions $f(\varphi)$ of the angle φ with period 2π can be expressed by the complex expansion coefficients $\mathring{f}(\nu)$ as

$$\mathcal{S}_\varphi\{f(\varphi)\} = \mathring{f}(\nu) = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)e^{-j\nu\varphi}\, d\varphi\ , \quad (5)$$

$$\mathcal{S}_\varphi^{-1}\{\mathring{f}(\nu)\} = f(\varphi) = \sum_{\nu=-\infty}^{\infty} \mathring{f}(\nu)e^{j\nu\varphi}\ . \quad (6)$$

Of special importance in this context are the following Fourier series expansions leading to Bessel functions

$$\mathcal{S}_\alpha\{e^{+jkr\cos(\theta-\alpha)}\} = j^\nu\, e^{-j\nu\theta}\, J_\nu(kr)\,, \quad (7)$$
$$\mathcal{S}_\theta\{e^{-jkr\cos(\theta-\alpha)}\} = j^{-\nu}\, e^{-j\nu\alpha}\, J_\nu(kr)\,. \quad (8)$$

2.3. Fourier-Bessel (Hankel) Transform

The ν-th order Fourier-Bessel transform (Hankel transform) of a function $f(r)$ for $r > 0$ is given by [2, 3]

$$\mathcal{H}_\nu\{f(r)\} = \hat{f}_\nu(k) = \int_0^\infty f(r)J_\nu(kr)\, r\, dr\ , \quad (9)$$

$$\mathcal{H}_\nu^{-1}\{\hat{f}_\nu(k)\} = f(r) = \int_0^\infty \hat{f}_\nu(k)J_\nu(kr)\, k\, dk\ . \quad (10)$$

2.4. Representations of one-dimensional signals

Figs. 1 and 2 compile graphical representations of the Fourier transforms with respect to time (1,2) and space (3,4), of the Fourier-Bessel transform (9,10), and of the Fourier series expansion (5,6). These relations will appear as building blocks of the corresponding multidimensional representations in Sec. 3.

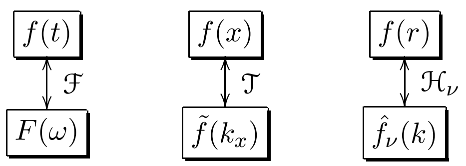

Figure 1: Representations of the Fourier transform $\mathcal{F}$ and $\mathcal{T}$ with respect to time and space domain (1–4) and of the Fourier-Bessel (Hankel) transform $\mathcal{H}_\nu$ (9,10)

3. MULTIDIMENSIONAL SIGNALS

The transforms for one-dimensional signals introduced in the previous section are now applied to multidimensional (MD) signals, specifically to signals which depend on time and two spatial coordinates. For the spatial coordinates, two different coordinate systems are considered, namely Cartesian and polar coordinates. Signals in polar coordinates may be expanded into a Fourier series with respect to the polar angle (angular expansion).

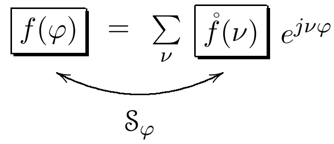

Figure 2: Representation of the Fourier series expansion $\mathcal{S}_\varphi$ ((5,6)

3.1. Time and space dependent signals

The Cartesian and polar coordinates are written as

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} r \\ \alpha \end{bmatrix} \quad (11)$$

respectively. The relations between Cartesian and polar coordinates are given by

$$\mathbf{x} = r\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \qquad \mathbf{r} = \begin{bmatrix} \sqrt{x^2+y^2} \\ \arctan\left(\frac{y}{x}\right) \end{bmatrix}\ . \quad (12)$$

Functions of time and of space in Cartesian coordinates are denoted by the subscript c as $f_\mathrm{c}(t, \mathbf{x})$. Similarly functions of time and of space in polar coordinates are written as $f_\mathrm{p}(t, \mathbf{r})$. The same spatial shape can be equally described in Cartesian and in polar coordinates, i.e. $f_\mathrm{c}(t, \mathbf{x}) = f_\mathrm{p}(t, \mathbf{r})$.

3.2. Fourier Transform with respect to time

The Fourier transforms of $f_\mathrm{c}(t, \mathbf{x})$ and $f_\mathrm{p}(t, \mathbf{r})$ with respect to time are denoted by $F_\mathrm{c}(\omega, \mathbf{x}) = \mathcal{F}\{f_\mathrm{c}(t, \mathbf{x})\}$ and $F_\mathrm{p}(\omega, \mathbf{r}) = \mathcal{F}\{f_\mathrm{p}(t, \mathbf{r})\}$.

3.3. Fourier Transform with respect to space

The Fourier transform with respect to space takes different forms for Cartesian and polar coordinates. They are denoted by $\mathcal{T}_\mathrm{c}$ and $\mathcal{T}_\mathrm{p}$, respectively.

The Fourier transform $\mathcal{T}_\mathrm{c}\{f_\mathrm{c}(t, \mathbf{x})\}$ with respect to space is defined by

$$\tilde{f}_{\mathrm{c}}(t,\mathbf{k}) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f_{\mathrm{c}}(t,\mathbf{x})e^{-j(\mathbf{x},\mathbf{k})}\,dx\,dy\,, \tag{13}$$

$$f_{\mathrm{c}}(t,\mathbf{x}) = \frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \tilde{f}_{\mathrm{c}}(t,\mathbf{k})e^{j(\mathbf{x},\mathbf{k})}\,dk_x\,dk_y\,. \tag{14}$$

$\mathbf{k} = [k_x, k_y]^T$ is the vector of spatial angular frequencies (wave numbers) and $(\mathbf{x},\mathbf{k})$ is the scalar product

$$(\mathbf{x},\mathbf{k}) = \mathbf{k}^T\mathbf{x} = k_x x + k_y y\,. \tag{15}$$

The Fourier transform pair for polar coordinates is obtained from the Cartesian version in (13,14) by substitution of the Cartesian variables $\mathbf{x}$ by the polar variables $\mathbf{r}$ according to (12). With a polar version $\mathbf{p}$ of the vector $\mathbf{k}$

$$\mathbf{p} = \begin{bmatrix} k \\ \theta \end{bmatrix}, \quad k = |\mathbf{k}|, \quad \tan\theta = \frac{k_y}{k_x}\,, \tag{16}$$

the spatial Fourier transform $\mathcal{T}_{\mathrm{p}}\{f_{\mathrm{p}}(t,\mathbf{r})\}$ in polar coordinates becomes

$$\tilde{f}_{\mathrm{p}}(t,\mathbf{p}) = \int\limits_{0}^{2\pi}\int\limits_{0}^{\infty} f_{\mathrm{p}}(t,\mathbf{r})e^{-j<\mathbf{r},\mathbf{p}>}\,r\,dr\,d\alpha \tag{17}$$

$$f_{\mathrm{p}}(t,\mathbf{r}) = \frac{1}{4\pi^2}\int\limits_{0}^{2\pi}\int\limits_{0}^{\infty} \tilde{f}_{\mathrm{p}}(t,\mathbf{p})e^{j<\mathbf{r},\mathbf{p}>}\,k\,dk\,d\theta. \tag{18}$$

The scalar product (15) becomes in polar coordinates

$$(\mathbf{x},\mathbf{k}) = kr\cos(\theta-\alpha) = <\mathbf{r},\mathbf{p}>\,. \tag{19}$$

3.4. Two-Dimensional Dirac Impulses

The inverse Fourier transform $\mathcal{T}_{\mathrm{c}}^{-1}$ defines for $\tilde{f}_{\mathrm{c}}(t,\mathbf{k}) \equiv 1$ the two-dimensional Dirac impulse [4] in Cartesian coordinates $\delta_{\mathrm{c}}(\mathbf{x}-\mathbf{x}_0)$

$$\delta_{\mathrm{c}}(\mathbf{x}-\mathbf{x}_0) = \delta(x-x_0)\delta(y-y_0) = $$
$$= \frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{j(\mathbf{x}-\mathbf{x}_0,\mathbf{k})}\,dk_x\,dk_y\,. \tag{20}$$

$\delta(x)$ denotes the one-dimensional Dirac impulse.

By substitution of the Cartesian with polar coordinates follows the two-dimensional Dirac impulse in polar coordinates

$$\delta_{\mathrm{p}}(\mathbf{r}-\mathbf{r}_0) = \frac{1}{r_0}\delta(r-r_0)\delta(\alpha-\alpha_0)\,. \tag{21}$$

When $\mathbf{x}_0$ and $\mathbf{r}_0$ are related as in (11, 12) then $\delta_{\mathrm{c}}(\mathbf{x}-\mathbf{x}_0) = \delta_{\mathrm{p}}(\mathbf{r}-\mathbf{r}_0)$.

Dirac impulses in the frequency domain for Cartesian and polar coordinates are defined by

$$\delta_{\mathrm{c}}(\mathbf{k}-\mathbf{k}_0) = \delta(k_x-k_{0,x})\delta(k_y-k_{0,y})\,, \tag{22}$$

$$\delta_{\mathrm{p}}(\mathbf{p}-\mathbf{p}_0) = \frac{1}{k_0}\delta(k-k_0)\delta(\theta-\theta_0)\,. \tag{23}$$

3.5. Multidimensional Fourier Transform

Application of the Fourier transforms with respect to time and space defines a multidimensional Fourier transform denoted by $\mathcal{FT}$

$$\tilde{F}_{\mathrm{c}}(\omega,\mathbf{k}) = \mathcal{FT}_{\mathrm{c}}\{f_{\mathrm{c}}(t,\mathbf{x})\} = \mathcal{T}_{\mathrm{c}}\{\mathcal{F}\{f_{\mathrm{c}}(t,\mathbf{x})\}\} \tag{24}$$

$$\tilde{F}_{\mathrm{p}}(\omega,\mathbf{p}) = \mathcal{FT}_{\mathrm{p}}\{f_{\mathrm{p}}(t,\mathbf{r})\} = \mathcal{T}_{\mathrm{p}}\{\mathcal{F}\{f_{\mathrm{p}}(t,\mathbf{r})\}\} \tag{25}$$

3.6. Angular Expansions

The functions $F_{\mathrm{p}}(\omega,\mathbf{r})$ and $\tilde{F}_{\mathrm{p}}(\omega,\mathbf{p})$ are periodic in α and in θ, respectively. Therefore, they can be expanded into Fourier series with respect to α and θ using the properties (7,8)

$$\mathring{F}_{\mathrm{p}}(\omega,r,\nu) = \mathcal{S}_\alpha\{F_{\mathrm{p}}(\omega,\mathbf{r})\}\,, \tag{26}$$

$$\mathring{\tilde{F}}_{\mathrm{p}}(\omega,k,\nu) = \mathcal{S}_\theta\{\tilde{F}_{\mathrm{p}}(\omega,\mathbf{p})\}\,. \tag{27}$$

Then $\mathring{\tilde{F}}_{\mathrm{p}}(\omega,k,\nu)$ can be expressed as

$$\mathring{\tilde{F}}_{\mathrm{p}}(\omega,k,\nu) = \mathcal{S}_\theta\{\mathcal{T}_{\mathrm{p}}\{F_{\mathrm{p}}(\omega,\mathbf{r})\}\} = \tag{28}$$
$$= \int\limits_{0}^{\infty}\int\limits_{0}^{2\pi} F_{\mathrm{p}}(\omega,\mathbf{r})\,\mathcal{S}_\theta\{e^{-j<\mathbf{p},\mathbf{r}>}\}\,d\alpha\,r\,dr\,.$$

With (8) and (26) follows a relation between the angular expansions $\mathring{F}_{\mathrm{p}}(\omega,r,\nu)$ and $\mathring{\tilde{F}}_{\mathrm{p}}(\omega,k,\nu)$

$$\mathring{\tilde{F}}_{\mathrm{p}}(\omega,k,\nu) = \int\limits_{0}^{\infty} \mathring{F}_{\mathrm{p}}(\omega,r,\nu)\,\frac{2\pi}{j^\nu}\,J_\nu(kr)\,r\,dr\,. \tag{29}$$

Since this relation bears close resemblance with the Hankel transform (9), it is called a *modified Hankel transform* $\overline{\mathcal{H}}_\nu$. An approach analogous to (28) for

$$\mathring{F}_{\mathrm{p}}(\omega,r,\nu) = \mathcal{S}_\alpha\{\mathcal{T}_{\mathrm{p}}^{-1}\{\tilde{F}_{\mathrm{p}}(\omega,\mathbf{p})\}\}$$

leads to a relation inverse to (29). Thus the modified Hankel transform $\overline{\mathcal{H}}_\nu\{\mathring{F}_{\mathrm{p}}(\omega,r,\nu)\} = \mathring{\tilde{F}}_{\mathrm{p}}(\omega,k,\nu)$ is given by

$$\mathring{\tilde{F}}_\mathrm{p}(\omega,k,\nu) = \int\limits_0^\infty \mathring{F}_\mathrm{p}(\omega,r,\nu)\,\frac{2\pi}{j^\nu}\,J_\nu(kr)\,r\,dr\ , \quad (30)$$

$$\mathring{F}_\mathrm{p}(\omega,r,\nu) = \int\limits_0^\infty \mathring{\tilde{F}}_\mathrm{p}(\omega,k,\nu)\,\frac{j^\nu}{2\pi}\,J_\nu(kr)\,k\,dk\ . \quad (31)$$

3.7. Representations of Multidimensional Signals

The sections above have developed the relations between MD signals in Cartesian and polar coordinates and in various transform domains. The resulting representation of MD signals and the relations between them are compiled in Figs. 3 and 4 ($\Longleftrightarrow$ denotes conversion between Cartesian and polar coordinates.)

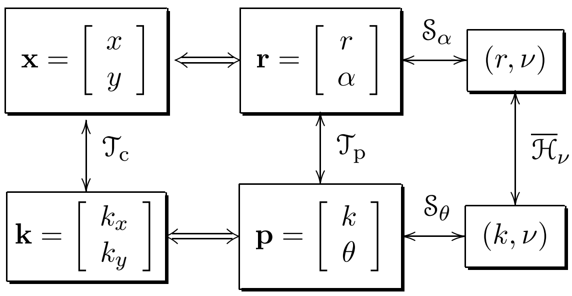

Figure 3: Coordinates in the space domain and in the spatial frequency domain in different coordinate systems.

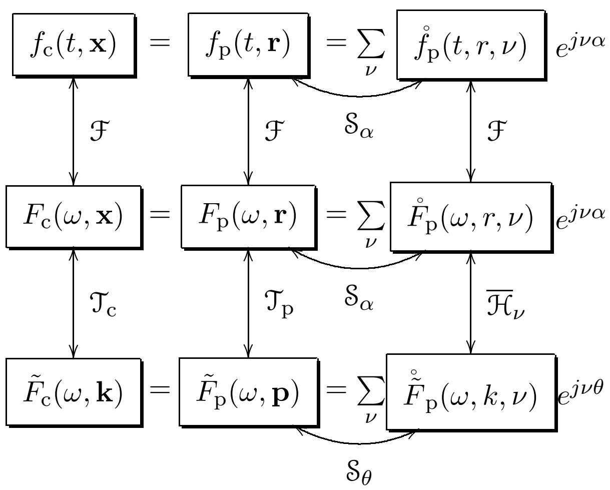

Figure 4: Representations of MD signals in the time and space domain and in the corresponding frequency domains in different coordinate systems

4. WAVE FIELDS

After having introduced various representations of general MD signals, the results are specialized to a certain class of signals. From now on only those signals are considered which satisfy the wave equation. This restriction poses strong constraints on the admissible signals by closely relating their variations in time and space. The formulations for these constraints depend on the chosen domain and are explored in the sequel.

Signals which satisfy the wave equation are also called solutions of the wave equation or wave fields or simply waves. They are denoted by $p_\mathrm{c}(t,\mathbf{x})$ or by the corresponding notations according to Fig. 4. First, signals in Cartesian coordinates will be considered, then follow polar coordinates and Fourier series expansions.

4.1. The Wave Equation

The *wave equation* is given by

$$\Delta p_\mathrm{c}(t,\mathbf{x}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\,p_\mathrm{c}(t,\mathbf{x}) = 0\ . \quad (32)$$

$\Delta = \nabla^2$ denotes the Laplace operator [5]. Application of the Fourier transform $\mathcal{F}$ with respect to time turns the wave equation into the *Helmholtz equation*

$$\Delta P_\mathrm{c}(\omega,\mathbf{x}) + \left(\frac{\omega}{c}\right)^2\,P_\mathrm{c}(\omega,\mathbf{x}) = 0\ . \quad (33)$$

Application of the Fourier transform $\mathcal{T}_\mathrm{c}$ with respect to space leads to a multiplication with $(j\mathbf{k})^T(j\mathbf{k}) = -k^2$

$$-k^2\tilde{P}_\mathrm{c}(\omega,\mathbf{k}) + \left(\frac{\omega}{c}\right)^2\,\tilde{P}_\mathrm{c}(\omega,\mathbf{k}) = 0\ . \quad (34)$$

In the spatial and temporal frequency domain, the constraint on the possible solutions of the wave equation consists of a strong coupling between the temporal frequency variable ω and the magnitude k of the spatial frequency variable $\mathbf{k}$, i.e.

$$\tilde{P}_\mathrm{c}(\omega,\mathbf{k}) = 0 \quad \text{for} \quad k^2 \neq \frac{\omega^2}{c^2}\ .$$

The following sections explain how this constraints affect the various representations from Fig. 4.

4.2. Plane Wave Solution of the Wave Equation

A *plane wave* is a special solution of the wave equation, which has a very simple form for Cartesian coordinates. It is determined by its waveform and by the direction from which the waveform emanates. The waveform is given by a time function $g_0(t)$ and the direction is given by the unit vector $\mathbf{n_0}$

$$\mathbf{n_0} = \begin{bmatrix} n_{0,x} \\ n_{0,y} \end{bmatrix} = \begin{bmatrix} \cos\theta_0 \\ \sin\theta_0 \end{bmatrix} . \qquad (35)$$

The plane wave solution is the MD signal

$$p_{c,0}(t, \mathbf{x}) = g_0 \left(t + \frac{1}{c}(\mathbf{x}, \mathbf{n_0}) \right) , \qquad (36)$$

where $(\mathbf{x}, \mathbf{n_0})$ is the scalar product between $\mathbf{x}$ and $\mathbf{n_0}$. It describes a planar wave front which propagates through space from the direction of $\mathbf{n_0}$ with speed c. In the origin $\mathbf{x} = \mathbf{0}$, the wave form $g_0(t)$ is observed directly as $p_{c,0}(t, \mathbf{0}) = g_0(t)$.

Now the Fourier transforms $\mathcal{F}$ and $\mathcal{T}_c$ with respect to time and space are applied to the plane wave solution. First, Fourier transform $\mathcal{F}$ with respect to time gives

$$P_{c,0}(\omega, \mathbf{x}) = G_0(\omega) e^{j\frac{\omega}{c}(\mathbf{x},\mathbf{n_0})} = G_0(\omega) e^{j(\mathbf{x},\mathbf{k_0})} \quad (37)$$

where $\mathbf{k}_0$ is the spatial frequency vector in the direction $\mathbf{n_0}$ from where the plane wave is emitted.

$$\mathbf{k}_0 = \frac{\omega}{c}\mathbf{n_0} = k_0 \mathbf{n_0} = k_0 \begin{bmatrix} n_{0,x} \\ n_{0,y} \end{bmatrix} = \begin{bmatrix} k_{0,x} \\ k_{0,y} \end{bmatrix} ,$$

$$k_{0,x}^2 + k_{0,y}^2 = k_0^2 = \frac{\omega^2}{c^2} . \qquad (38)$$

Next, spatial Fourier transform $\mathcal{T}_c$ gives with (22)

$$\tilde{P}_{c,0}(\omega, \mathbf{k}) = 4\pi^2 G_0(\omega) \delta_c(\mathbf{k} - \mathbf{k}_0) . \qquad (39)$$

Equation (39) shows that temporal and spatial frequency variables in $\tilde{P}_{c,0}(\omega, \mathbf{k})$ are closely related. Waves have a spatial spectrum which is rather restricted, i.e. it is a 2D Dirac pulse in the k_x–k_y–plane. The distance from the origin is given by the temporal frequency ω due to $|\mathbf{k}_0| = \omega/c$. The direction is determined by the direction of wave emanation $\mathbf{n_0}$. Fig. 5 shows the possible locations of $\delta_c(\mathbf{k} - \mathbf{k}_0)$.

4.3. General Solution of the Wave Equation

The wave equation admits more general solutions than the plane wave solution discussed above. However, the general solution may be obtained by superimposing plane waves from all possible directions ($0 \leq \theta_0 < 2\pi$) and with varying waveforms. To express the dependency on the angle θ_0 properly, the plane wave solution (39) is rewritten as

$$\tilde{P}_{c,0}(\omega, \mathbf{k}) = \tilde{P}_c(\omega, \theta_0, \mathbf{k}) = 4\pi^2 G(\omega, \theta_0) \delta_c(\mathbf{k} - \mathbf{k}_0) . \qquad (40)$$

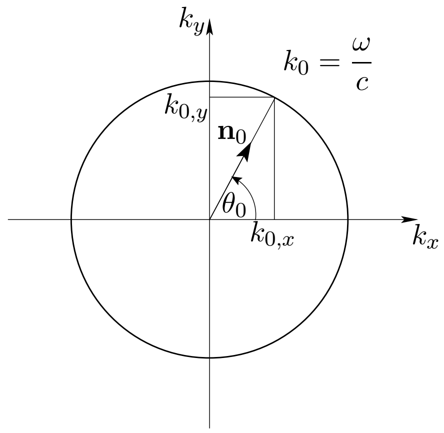

Figure 5: Possible locations of the Dirac impulse $\delta_c(\mathbf{k} - \mathbf{k}_0)$ in the k_x–k_y–plane

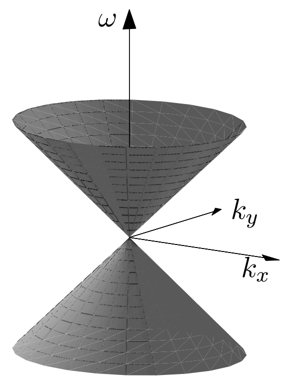

Figure 6: Support of the Dirac impulse $\delta_c(\mathbf{k} - \mathbf{k}_0)$ in the (ω, k_x, k_y)–domain

The general solution is obtained by integration over all possibe directions θ_0

$$\tilde{P}_c(\omega, \mathbf{k}) = \int_0^{2\pi} \tilde{P}_c(\omega, \theta_0, \mathbf{k}) \, d\theta_0 =$$

$$= 4\pi^2 \int_0^{2\pi} G(\omega, \theta_0) \delta_c(\mathbf{k} - \mathbf{k}_0) \, d\theta_0 . \qquad (41)$$

This construction of the general solution is straightforward, but it is not compatible with either a Cartesian or a polar coordinate system. While the spatial frequency vector $\mathbf{k}$ is formulated in Cartesian coordinates, the angle dependence in $G(\omega, \theta_0)$ corresponds to polar coordinates. In the sequel, equation (41) is expressed first in Cartesian and then in polar coordinates.

4.3.1. Cartesian coordinates

To express (41) in Cartesian coordinates, the angle θ_0 has to be substituted by $k_{0,x}$ (or $k_{0,y}$). It is of advantage to break the integral from 0 to 2π into two integrals from 0 to π and π to 2π. Introducing $\tilde{G}^{(1)}(\omega, k_{0,x})$ for $0 \leq \theta_0 < \pi$ and $\tilde{G}^{(2)}(\omega, k_{0,x})$ for $\pi \leq \theta_0 < 2\pi$

$$\tilde{G}^{(1)}(\omega, k_{0,x}) = \begin{cases} G(\omega, \theta_0) & |k_{0,x}| \leq k_0 \\ 0 & |k_{0,x}| > k_0 \end{cases} , \quad (42)$$

$$\tilde{G}^{(2)}(\omega, k_{0,x}) = \begin{cases} -G(\omega, \theta_0) & |k_{0,x}| \leq k_0 \\ 0 & |k_{0,x}| > k_0 \end{cases} , \quad (43)$$

and following the standard procedures for the substitution in integrals leads to

$$\tilde{P}_{\mathrm{c}}(\omega, \mathbf{k}) = \tilde{P}_{\mathrm{c}}^{(1)}(\omega, \mathbf{k}) + \tilde{P}_{\mathrm{c}}^{(2)}(\omega, \mathbf{k}) , \quad (44)$$

with (for $i = 1, 2$)

$$\tilde{P}_{\mathrm{c}}^{(i)}(\omega, \mathbf{k}) = 4\pi^2 \int_{-\infty}^{\infty} \tilde{G}^{(i)}(\omega, k_{0,x}) \frac{1}{\sqrt{k_0^2 - k_{0,x}^2}} \delta_{\mathrm{c}}(\mathbf{k} - \mathbf{k}_0)\, dk_{0,x}. \quad (45)$$

The two terms in (44) have a physical interpretation as waves emanating from opposite directions. Since both terms have an identical structure, they will not be further distinguished from now on. The results derived below apply to either term with proper definition of $\tilde{G}(\omega, k_{0,x})$ according to (42) or (43).

Equation (45) shows that the 2D spatial Fourier transform $\tilde{P}_{\mathrm{c}}(\omega, \mathbf{k})$ is generated by a signal with one spatial dimension only. Denoting the Fourier transform of this signal by

$$\tilde{H}_{\mathrm{c}}(\omega, k_{0,x}) = \tilde{G}(\omega, k_{0,x}) \frac{2\pi}{\sqrt{\left(\frac{\omega}{c}\right)^2 - k_{0,x}^2}} \quad (46)$$

gives equation (45) the concise form (the superscript (i) now omitted)

$$\tilde{P}_{\mathrm{c}}(\omega, \mathbf{k}) = 2\pi \int_{-\infty}^{\infty} \tilde{H}_{\mathrm{c}}(\omega, k_{0,x}) \delta_{\mathrm{c}}(\mathbf{k} - \mathbf{k}_0)\, dk_{0,x}. \quad (47)$$

To show the relation between $\tilde{P}_{\mathrm{c}}(\omega, \mathbf{k})$ and its spatially 1D counterpart $\tilde{H}_{\mathrm{c}}(\omega, k_{0,x})$, the definition of the Dirac impulse (22) and the dispersion relation according to (38) are used

$$\delta_{\mathrm{c}}(\mathbf{k} - \mathbf{k}_0) = \delta(k_x - k_{0,x})\, \delta\left(k_y - \sqrt{\left(\frac{\omega}{c}\right)^2 - k_{0,x}^2}\right) . \quad (48)$$

Performing the 1D integration in (47) yields

$$\tilde{P}_{\mathrm{c}}(\omega, \mathbf{k}) = 2\pi\, \tilde{H}_{\mathrm{c}}(\omega, k_x)\, \delta\left(k_y - \sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2}\right) . \quad (49)$$

This representation shows, that the spatially 2D spectrum $\tilde{P}_{\mathrm{c}}(\omega, \mathbf{k})$ exists only on the region of support in the (ω, k_x, k_y)–domain indicated in Fig. 6.

The inverse transformation $\mathcal{T}_{\mathrm{c}}^{-1}$ with respect to $\mathbf{k}$ gives

$$P_{\mathrm{c}}(\omega, \mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{H}_{\mathrm{c}}(\omega, k_x) e^{j\left(k_x x + \sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2}\, y\right)}\, dk_x . \quad (50)$$

Equation (50) establishes the connection between a general solution of the wave equation $P_{\mathrm{c}}(\omega, \mathbf{x})$ and the 1D spatial Fourier spectrum $\tilde{H}_{\mathrm{c}}(\omega, k_x)$. This relation can be understood more clearly, if $\tilde{P}_{\mathrm{c}}(\omega, \mathbf{x})$ is restricted to its values on the x-axis, i.e. $y = 0$. Then (50) turns into a one-dimensional inverse Fourier transform with respect to space according to (4)

$$P_{\mathrm{c}}(\omega, \mathbf{x})\big|_{y=0} = H_{\mathrm{c}}(\omega, x) = \mathcal{T}^{-1}\{\tilde{H}_{\mathrm{c}}(\omega, k_x)\} , \quad (51)$$

The relations shown above give rise to the following interpretations:

From (51) follows

$$\tilde{H}_{\mathrm{c}}(\omega, k_x) = \mathcal{T}\{H_{\mathrm{c}}(\omega, x)\} = \mathcal{T}\{\, P_{\mathrm{c}}(\omega, \mathbf{x})\big|_{y=0}\}. \quad (52)$$

This relation along with (50) shows that a spatially 2D wave field is completely determined by its values along a line in the x-y plane, i.e. a spatially 1D signal. For the derivation given here, this line is the x-axis, but by suitable coordinate transformations the representation could be extended to other lines in the x-y plane which are not orthogonal to $\mathbf{n}_0$.

From (46) follows

$$\tilde{G}(\omega, k_x) = \frac{1}{2\pi}\, \tilde{H}_{\mathrm{c}}(\omega, k_x) \sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2} , \quad (53)$$

The values along the x-axis are related by (53) to the Fourier spectrum of the waveform of plane waves from different angles of incidence. This relation can be interpreted as a projection of the waveform spectra $\tilde{G}(\omega, k_x)$ to the k_x-axis of the Cartesian coordinate system.

Equation (49) shows that the frequency domain representation $\tilde{P}(\omega, \mathbf{k})$ of a wave field is nonzero only on a cone in the (ω, k_x, k_y)–domain as shown in Fig. 6.

The relations developed in this subsection are the specialization of the the 2D Fourier transform $\mathcal{T}_\mathrm{c}$ for general MD signals to solutions of the wave equation. They are also known as *plane wave expansion* or *plane wave decomposition* (see e.g. [6]).

4.3.2. Polar coordinates

To formulate these results in polar coordinates, the general solution of the wave equation in the form of (41) is reconsidered. Converting it from Cartesian to polar coordinates results in [1]

$$\tilde{P}_\mathrm{p}(\omega, \mathbf{p}) = 4\pi^2 \int_0^{2\pi} G(\omega, \theta_0)\delta_\mathrm{p}(\mathbf{p} - \mathbf{p}_0)\, d\theta_0 \ . \quad (54)$$

Observing (23) and carrying out the integration in (54) yields

$$\tilde{P}_\mathrm{p}(\omega, \mathbf{p}) = 4\pi^2\, G(\omega, \theta)\, \frac{1}{k}\, \delta \left(k - \frac{\omega}{c}\right) \ . \quad (55)$$

As in the Cartesian case the frequency domain representation $\tilde{P}_\mathrm{p}(\omega, \mathbf{p})$ is nonzero only on a cone in the (ω, k_x, k_y)–domain or in the (ω, k, θ)–domain respectively (see Fig. 6). This condition is mathematically represented by the Dirac impulse incorporating the dispersion relation in (55).

By inverse Fourier transformation $P_\mathrm{p}(\omega, \mathbf{r}) = \mathcal{T}_\mathrm{p}^{-1}\{\tilde{P}_\mathrm{p}(\omega, \mathbf{p})\}$ with respect to space follows

$$P_\mathrm{p}(\omega, \mathbf{r}) = \int_0^{2\pi} G(\omega, \theta) e^{j\frac{\omega}{c} r \cos(\theta - \alpha)}\, d\theta \ . \quad (56)$$

As for Cartesian coordinates in (50), the wave field in polar coordinates is generated by a spatially 1D signal. Here, this signal is given by the waveform spectrum $G(\omega, \theta)$.

4.3.3. Angular Expansions

The Fourier coeffients $\mathring{P}_\mathrm{p}(\omega, r, \nu)$ of $P_\mathrm{p}(\omega, \mathbf{r})$ and $\mathring{\tilde{P}}_\mathrm{p}(\omega, k, \nu)$ of $\tilde{P}_\mathrm{p}(\omega, \mathbf{p})$ follow from a Fourier series expansion of the wave field description in polar coordinates according to (54) and (56). The coefficients $\mathring{P}_\mathrm{p}(\omega, r, \nu) = \mathcal{S}_\alpha\{P_\mathrm{p}(\omega, \mathbf{r})\}$ are given by

$$\mathring{P}_\mathrm{p}(\omega, r, \nu) = \int_0^{2\pi} G(\omega, \theta_0)\, \mathcal{S}_\alpha\{e^{j\frac{\omega}{c} r \cos(\theta_0 - \alpha)}\}\, d\theta_0 \ . \quad (57)$$

With (7) follows

$$\mathring{P}_\mathrm{p}(\omega, r, \nu) = 2\pi j^\nu J_\nu \left(\frac{\omega}{c} r\right) \mathring{G}(\omega, \nu) \quad (58)$$

with the Fourier coefficients of $G(\omega, \theta)$

$$\mathring{G}(\omega, \nu) = \mathcal{S}_\theta\{G(\omega, \theta)\} = \frac{1}{2\pi} \int_0^{2\pi} G(\omega, \theta) e^{-j\nu\theta}\, d\theta \ . \quad (59)$$

The Fourier coefficients $\mathring{P}_\mathrm{p}(\omega, r, \nu)$ of the spatially 2D signal $P_\mathrm{p}(\omega, \mathbf{r})$ can be represented in terms of the Fourier coefficients $\mathring{G}(\omega, \nu)$ of the spatially 1D signal $G(\omega, \theta)$.

There are two ways to calculate the coefficients $\mathring{\tilde{P}}_\mathrm{p}(\omega, k, \nu)$ of $\tilde{P}_\mathrm{p}(\omega, \mathbf{p})$ (see Fig. 4). One way is to apply $\mathcal{S}_\theta$ directly to $\tilde{P}_\mathrm{p}(\omega, \mathbf{p})$. The second way is to apply the modified Hankel transform $\overline{\mathcal{H}}_\nu$ to the Fourier coefficients $\mathring{P}_\mathrm{p}(\omega, r, \nu)$. In either case, the result is [1]

$$\mathring{\tilde{P}}_\mathrm{p}(\omega, k, \nu) = 4\pi^2\, \mathring{G}(\omega, \nu)\, \frac{1}{k} \delta \left(k - \frac{\omega}{c}\right) \ . \quad (60)$$

Again, the Fourier coefficients $\mathring{\tilde{P}}_\mathrm{p}(\omega, k, \nu)$ of the spatially 2D signal $\tilde{P}_\mathrm{p}(\omega, \mathbf{p})$ can be represented in terms of the Fourier coefficients $\mathring{G}(\omega, \nu)$.

4.4. Compilation of the Results

The properties of MD signals which describe wave fields are now compiled in concise form. The contents of table 1 specify the properties of the general signals from Fig. 4 for the case that they satisfy the wave equation. The spatially two-dimensional signals which solve the wave equation can be represented by a spatially one-dimensional quantity. This statement holds for all considered coordinate systems and for both space and spatial frequency domains. This one-dimensional quantity can be derived directly from the general solution constituted by the wave forms of plane waves from all directions. These wave forms appear either directly as their temporal Fourier transform $G(\omega, \theta)$, its Fourier coefficients $\mathring{G}(\omega, \nu)$, or as the projection $\tilde{H}_\mathrm{c}(\omega, k_x)$ of $G(\omega, \theta)$ to the k_x–axis of the coordinate system of the spatial frequency domain. This projection corresponds to the values $H_\mathrm{c}(\omega, x)$ of the wave field $P_\mathrm{c}(\omega, \mathbf{x})$ along the x–axis.

5. CONCLUSIONS

A framework for the representation of two-dimensional signals in different coordinate systems and different

Cartesian Coordinates	
$P_{\mathrm{c}}(\omega,\mathbf{x}) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \tilde{H}_{\mathrm{c}}(\omega,k_x) e^{j\left(k_x x + \sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2}\; y\right)}\, dk_x$	$\tilde{P}_{\mathrm{c}}(\omega,\mathbf{k}) = 2\pi\,\delta\left(k_y - \sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2}\right)\tilde{H}_{\mathrm{c}}(\omega,k_x)$
$P_{\mathrm{c}}(\omega,\mathbf{x})\big\|_{y=0} = H_{\mathrm{c}}(\omega,x)$	$\tilde{G}(\omega,k_x) = \frac{1}{2\pi}\,\tilde{H}_{\mathrm{c}}(\omega,k_x)\sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2}$
Polar Coordinates	
$P_{\mathrm{p}}(\omega,\mathbf{r}) = \int\limits_{0}^{2\pi} G(\omega,\theta) e^{j\frac{\omega}{c} r\cos(\theta-\alpha)}\, d\theta$	$\tilde{P}_{\mathrm{p}}(\omega,\mathbf{p}) = 4\pi^2\, G(\omega,\theta)\,\frac{1}{k}\,\delta\left(k - \frac{\omega}{c}\right)$
Fourier Series Coefficients	
$\mathring{P}_{\mathrm{p}}(\omega,r,\nu) = 2\pi j^{\nu} J_{\nu}\left(\frac{\omega}{c} r\right)\mathring{G}(\omega,\nu)$	$\mathring{\tilde{P}}_{\mathrm{p}}(\omega,k,\nu) = 4\pi^2\,\mathring{G}(\omega,\nu)\,\frac{1}{k}\,\delta\left(k - \frac{\omega}{c}\right)$

Table 1: Representation of a wave field in the space domain (left) and in the spatial frequency domain (right)

transform domains has been developed. It comprises known results for wave fields as usually derived in the fields of imaging and acoustics. By adopting a general viewpoint from the theory of signals and systems, these results could be derived in a straightforward fashion and without resorting to special applications.

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