Audio Engineering Society

Convention Paper

Presented at the 126th Convention
2009 May 7–10 Munich, Germany

Sound Field Reproduction Employing Non-Omnidirectional Loudspeakers

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ABSTRACT

In this paper we treat sound field reproduction via circular distributions of loudspeakers. The general formulation of the approach has been recently published by the authors. In this contribution, we concentrate on the employment of secondary sources (i.e. loudspeakers) whose spatio-temporal transfer function is not omnidirectional. The presented approach allows to treat each spatial mode of the secondary source's spatio-temporal transfer function individually. We finally outline the general process of incorporating spatio-temporal transfer functions obtained from microphone array measurements.

1. INTRODUCTION

Traditionally, massive-multichannel sound field reproduction approaches like wave field synthesis or higher order Ambisonics assume that the involved secondary sources (i.e. loudspeakers) are omnidirectional. For lower frequencies, this assumption is indeed approximately fulfilled when conventional loudspeakers with closed cabinets are considered. However, for higher frequencies above a few thousand Hertz complex radiation patterns evolve.

A number of approaches based on the theory of multiple-input-multiple-output (MIMO) systems have been proposed in order to compensate for the influence of the reproduction room and the loudspeaker radiation characteristics [1, 2, 3, 4, 5]. Room compensation requires realtime analysis of the reproduced wave field and adaptive algorithms due to the time-variance of room acoustics (e.g. temperature variations [6]). Compensation of the loudspeaker radiation characteristics such as directivity and frequency response is less complex since it can be assumed that these characteristics are time-invariant. No adaptation and therefore no real-time analysis is required. However, in order that the radiation characteristics can be compensated for excluding the reproduction room, the radiation characteristics of the entire secondary source setup have to be measured under anechoic conditions. When certain physical constraints are accepted, a significant reduction of complexity can be achieved and a continuous formulation of the MIMO approaches can be established. Besides time-invariance, the fundamental physical constraints introduced in the presented approach are:

(1) The secondary source arrangement is circular.

(2) The spatio-temporal transfer function of the secondary sources is rotation invariant. In other words, all individual loudspeakers have to have equal radiation characteristics and have to be orientated towards the center of the secondary source setup.

Requirement (1) can obviously be fulfilled. Preliminary measurements undertaken at Deutsche Telekom Laboratories have shown that typical commercially available loudspeakers with closed cabinets indeed exhibit similar to equal spatio-temporal transfer functions in anechoic condition and when only one model of loudspeakers is considered. This suggests that requirement (2) can also be fulfilled when the acoustical properties of the reproduction room are ignored.
The presented approach is actually not a compensation for deviations of the loudspeaker radiation characteristics from certain assumptions (e.g. omnidirectionality). It is rather such that the formulation of the approach allows for an explicit consideration thereof. However, in accordance with the literature we also speak of loudspeaker directivity compensation in conjunction with the presented approach.
The approach treated in this paper has been presented by the authors in [7, 8], whereby formulations were kept general. In this contribution, we investigate the properties of the approach in a purely two-dimensional scenario.

2. NOMENCLATURE

For convenience, we restrict our considerations to two spatial dimensions. This means in this context that a wave field under consideration is independent from one of the spatial coordinates, i.e. $P(x, y, z, \omega) = P(x, y, \omega)$. The two-dimensional position vector in Cartesian coordinates is given as $\mathbf{x} = [x \;\; y]^T$. The Cartesian coordinates are linked to the polar coordinates via $x = r \cos\alpha$ and $y = r \sin\alpha$. Refer to the coordinate system depicted in figure 1.

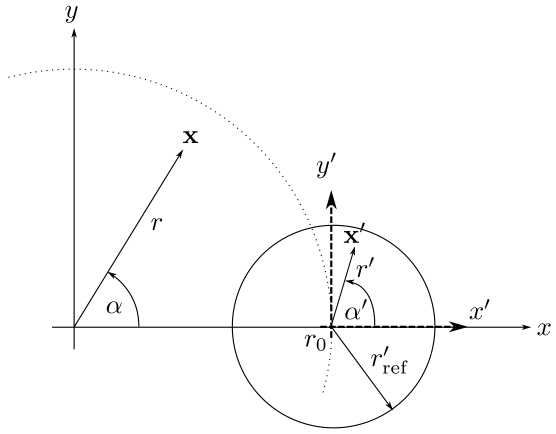


Fig. 1: The coordinate system used in this paper. The center of the secondary source distribution coincides with the origin of the global coordinate system. The dotted line indicates the secondary source distribution. The prime $'$ denotes quantities belonging to a local coordinate system with origin at $\mathbf{x}_0 = [r_0 \; 0]^T$ (refer to section 4).

The acoustic wavenumber is denoted by k. It is related to the temporal frequency by $k^2 = \left(\frac{\omega}{c}\right)^2$ with ω being the radial frequency and c the speed of sound. Outgoing monochromatic plane and cylindrical waves are denoted by $e^{-j\frac{\omega}{c}r\cos(\theta_{\text{pw}}-\alpha)}$ and $H_0^{(2)}(\frac{\omega}{c}r)$ respectively, with θ_{pw} being the propagation direction of the plane wave. The imaginary unit is denoted by j ($j = \sqrt{-1}$).

3. GENERAL FORMULATION

In this section, we briefly review the general approach presented by the authors in [7, 8]. Its physical fundament is the so-called *simple source approach* and it can be seen as an analytical formulation of what is known as higher order Ambisonics. The simple source approach for interior problems states that the acoustic field generated by events outside a volume can also be generated by a continuous distribution of secondary simple sources enclosing the respective volume [9].

As stated in section 2, we limit our derivations to two-dimensional reproduction for convenience. Furthermore, we assume the distribution of secondary sources to be circular. In order to fulfill the requirements of the simple source approach and therefore for artifact-free reproduction, the wave fields emitted by the secondary sources have to be two-dimensional. We thus have to assume a continuous circular distribution of secondary line sources positioned perpendicular to the target plane (the receiver plane) [9]. Our approach is therefore not directly implementable since loudspeakers exhibiting the properties of line sources are commonly not available. Real-world implementations usually employ loudspeakers with closed cabinets as secondary sources. The properties of these loudspeakers are more accurately modeled by point sources.
The main motivation to focus on two dimensions is to keep the mathematical formulation simple in order to illustrate the general principle of the presented approach. The extension both to three-dimensional reproduction (i.e. spherical arrays of secondary point sources) and to two-dimensional reproduction employing circular arrangements of secondary point sources (*2½-dimensional reproduction*) is straightforward and a general treatment thereof can be found e.g. in [7].

3.1. Derivation of the secondary source driving function

The reproduction equation for a continuous circular distribution of secondary line sources and with radius r_0 centered around the origin of the coordinate system is given by

$$P(\mathbf{x},\omega) = \int_0^{2\pi} D(\alpha_0,\omega)\ G_{\mathrm{2D}}(\mathbf{x}-\mathbf{x}_0,\omega)\ r_0\ d\alpha_0\ , \quad (1)$$

where $\mathbf{x}_0 = r_0 \cdot [\cos\alpha_0\ \sin\alpha_0]^T$. $P(\mathbf{x},\omega)$ denotes the reproduced wave field, $D(\alpha_0,\omega)$ the driving function for the secondary source situated at $\mathbf{x}_0$, and $G_{\mathrm{2D}}(\mathbf{x}-\mathbf{x}_0,\omega)$ its two-dimensional spatio-temporal transfer function. Note that we assume $G(\cdot)$ to be shift-invariant (we write $G(\mathbf{x}-\mathbf{x}_0,\omega)$ instead of $G(\mathbf{x}|\mathbf{x}_0,\omega)$) [9]. Refer also to section 4. See [10] for the general approach how to treat shift-variant systems.
A fundamental property of (1) is its inherent non-uniqueness and ill-posedness [11]. I.e. in certain situations, the solution is undefined and so-called *critical* or *forbidden frequencies* arise. The forbidden frequencies represent the resonances of the cavity under consideration. However, there are indications that the forbidden frequencies are only of minor relevance when practical implementations are considered [9].
Equation (1) constitutes a circular convolution and therefore the convolution theorem

$$\mathring{P}_\nu(r,\omega) = 2\pi r_0\ \mathring{D}_\nu(\omega)\ \mathring{G}_\nu(r,\omega) \quad (2)$$

applies [12]. $\mathring{P}_\nu(r,\omega)$, $\mathring{D}_\nu(\omega)$, and $\mathring{G}_\nu(r,\omega)$ denote the Fourier series expansion coefficients of $P(\mathbf{x},\omega)$, $D(\alpha,\omega)$, and $G_{\mathrm{2D}}\left(\mathbf{x}-[r_0\ 0]^T\right)$[1].
The Fourier series expansion coefficients $\mathring{F}_\nu(r,\omega)$ of a two-dimensional function $F(\mathbf{x},\omega)$ can be obtained via [9]

$$\mathring{F}_\nu(r,\omega) = \frac{1}{2\pi}\int_0^{2\pi} F(\mathbf{x},\omega)e^{-j\nu\alpha}\ d\alpha\ . \quad (3)$$

The function $F(\mathbf{x},\omega)$ can then be synthesized as

$$F(\mathbf{x},\omega) = \sum_{\nu=-\infty}^{\infty} \mathring{F}_\nu(r,\omega)\ e^{j\nu\alpha}\ . \quad (4)$$

For propagating wave fields the coefficients $\mathring{F}_\nu(r,\omega)$ can be decomposed as

$$\mathring{F}_\nu(r,\omega) = \breve{F}_\nu(\omega)\ J_\nu\left(\frac{\omega}{c}r\right)\ , \quad (5)$$

whereby $J_\nu(\cdot)$ denotes the ν-th order Bessel function [9]. For diverging wave fields the coefficients $\mathring{F}_\nu(r,\omega)$ can be decomposed as

$$\mathring{F}_\nu(r,\omega) = \breve{F}_\nu(\omega)\ H_\nu^{(2)}\left(\frac{\omega}{c}r\right)\ , \quad (6)$$

whereby $H_\nu^{(2)}(\cdot)$ denotes the ν-th order Hankel function of second kind [9].
From (2) and (5) we can deduce that

$$\mathring{D}_\nu(\omega) = \frac{1}{2\pi r_0}\ \frac{\mathring{P}_\nu(r,\omega)}{\mathring{G}_\nu(r,\omega)} = \quad (7)$$

$$= \frac{1}{2\pi r_0}\ \frac{\breve{P}_\nu(\omega)\cdot J_\nu\left(\frac{\omega}{c}r\right)}{\breve{G}_\nu(\omega)\cdot J_\nu\left(\frac{\omega}{c}r\right)}\ . \quad (8)$$

[1] Note that the coefficients $\mathring{G}_\nu(r,\omega)$ as used throughout this paper assume that the secondary source is situated at the position $(r = r_0, \alpha = 0)$ and is orientated towards the coordinate origin. Refer to section 4.

For $J_\nu\left(\frac{\omega}{c}r\right) \neq 0$ the Bessel functions in (8) cancel out directly. Wherever $J_\nu\left(\frac{\omega}{c}r\right) = 0$ de l'Hôpital's rule [13] can be applied to proof that the Bessel functions also cancel out in these cases, thus making $\mathring{D}_\nu(\omega)$ and therefore also $D(\alpha_0, \omega)$ independent from the receiver position.
Introducing the result into (4) finally yields the secondary source driving function $D(\alpha_0, \omega)$ for a secondary source situated at position $\mathbf{x}_0$ reproducing a desired wave field with expansion coefficients $\breve{P}_\nu(\omega)$ reading

$$D(\alpha, \omega) = \frac{1}{2\pi r_0} \sum_{\nu=-\infty}^{\infty} \frac{\breve{P}_\nu(\omega)}{\breve{G}_\nu(\omega)} e^{j\nu\alpha} \ , \qquad (9)$$

whereby we omitted the index 0 in α_0 for convenience. Note again that $D(\alpha, \omega)$ is independent from the receiver position.
$\breve{G}_\nu(\omega)$ describes the spatio-temporal transfer function of the involved secondary sources. $\breve{G}_\nu(\omega)$ are the coefficients as defined in (5) with respect to the expansion of $G(\mathbf{x}, \omega)$ around the origin of the global coordinate system.

3.2. Reproduced wave field

Equation (9) can be verified by inserting it into (1). After interchanging the order of integration and summation and exploitation of the orthogonality of the circular harmonics $e^{j\nu\alpha}$ [9], one arrives at the desired wave field, thus proving perfect reproduction apart from forbidden frequencies. Note however that the coefficients $\breve{P}_\nu(\omega)$ respectively $\breve{G}_\nu(\omega)$ are typically derived from interior expansions. This implies that the desired wave field is only correctly reproduced inside the secondary source distribution. We emphasize that in order to achieve this perfect reproduction, the secondary source distribution has to be continuous and the spatio-temporal transfer function of the secondary sources has to be two-dimensional. The latter means that the spatio-temporal transfer function of the secondary sources may not exhibit any variation in the vertical direction. This suggests the employment of line-like loudspeakers. The extension of the presented approach to three- and 2½-dimensional reproduction is straightforward and can be found in [7].

4. INCORPORATION OF THE LOUDSPEAKER DIRECTIVITY

Formulating (1) explicitly in polar coordinates as

$$P(r, \alpha, \omega) = \\ = \int_0^{2\pi} D(\alpha_0, \omega)\ G_{2\mathrm{D}}(r, r_0, \alpha - \alpha_0, \omega)\ r_0\ d\alpha_0 \ , \quad (10)$$

clearly reveals the convolution reading

$$P(r, \alpha, \omega) = D(\alpha, \omega) \circledast_\alpha G_{2\mathrm{D}}(r, r_0, \alpha - \alpha_0|_{\alpha_0=0}, \omega)\, r_0 \ , \quad (11)$$

whereby the asterisk $\circledast_\alpha$ indicates circular convolution with respect to α.
From (11) it becomes obvious that $\mathring{G}_\nu(r, \omega)$ in (2) are the Fourier expansion coefficients of the spatio-temporal transfer function of a secondary source situated at the position $(r = r_0,\ \alpha = 0)$ respectively $\mathbf{x} = [r_0\ 0]^T$. The expansion center is the center of the secondary source distribution, i.e. the origin of the coordinate system.
Furthermore, from (11) we can deduce that the spatio-temporal transfer functions of all secondary sources need to be invariant with respect to rotation around the center of the secondary source distribution. In other words, all individual loudspeakers have to have equal radiation characteristics and have to be orientated towards the center of the secondary source setup.
The spatio-temporal transfer function of loudspeakers is typically described via the coefficients $\mathring{G}'_\mu(r', \omega)$ (see below) of an expansion around the acoustical center of the loudspeaker (which is referred to as its *position*). We refer to $\mathring{G}'_\mu(r', \omega)$ as *secondary source directivity coefficients*. The secondary source directivity coefficients can be directly obtained from microphone array measurements, e.g. [14].
We assume that the loudspeaker under consideration is positioned at $\mathbf{x}_0 = [r_0\ 0]^T$ and is orientated towards the origin of the global coordinate system. We establish a local coordinate system with origin at $\mathbf{x}_0$ and whose axes are parallel to those of the global coordinate system (refer to figure 1). Quantities belonging to the local coordinate system are denoted with a prime $'$.
The spatio-temporal transfer function of the loud-

speaker under consideration is then given by

$$G(\mathbf{x}',\omega) = \sum_{\mu=-\infty}^{\infty} \breve{G}'_\mu(\omega)\ H_\mu^{(2)}\left(\frac{\omega}{c}r'\right)\ e^{j\mu\alpha'}\ . \quad (12)$$

We apply the harmonic addition theorem [15] in order to translate the center of the expansion to the center of the global coordinate system yielding

$$G(\mathbf{x},\omega) = \sum_{\nu=-\infty}^{\infty} e^{j\nu\alpha}\times \\ \times \underbrace{\sum_{\mu=-\infty}^{\infty} \breve{G}'_\mu(\omega)\ H_{\nu-\mu}^{(2)}\left(\frac{\omega}{c}r_0\right)}_{=\breve{G}_\nu(\omega)}\ J_\nu\left(\frac{\omega}{c}r\right)\ , \\ \forall\ r < r_0 \quad (13)$$

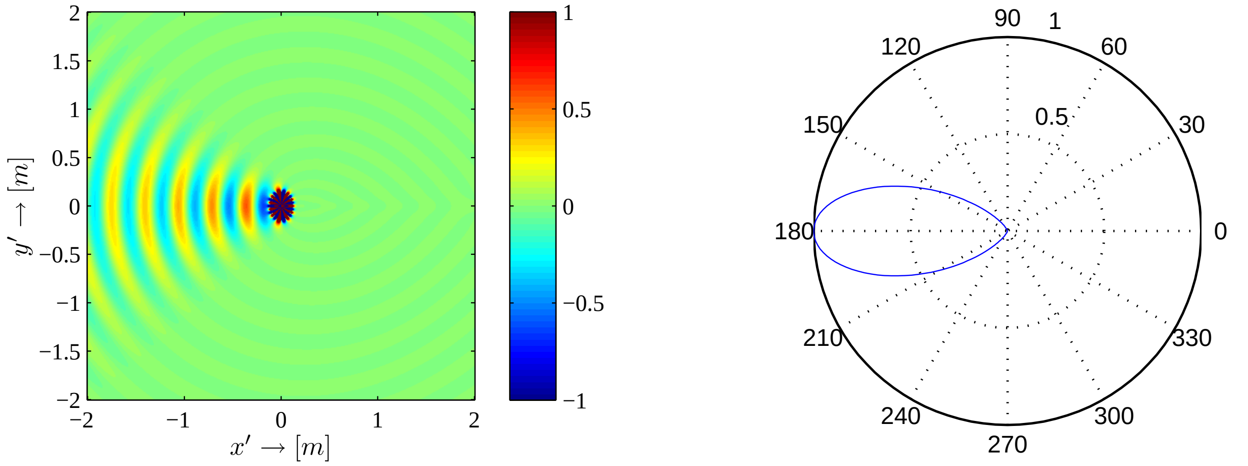
In (13) the coefficients $\breve{G}_\nu(\omega)$ to be inserted into the driving function (9) become apparent.
In order that the driving function (9) is defined each mode $\breve{G}_\nu(\omega)$ of the compensation filter may not exhibit zeros since $\breve{G}_\nu(\omega)$ appears in the denominator of the driving function (9). $\breve{G}_\nu(\omega)$ for a given order ν is given by a summation over the product of all loudspeaker directivity coefficients $\breve{G}'_\mu(\omega)$ and Hankel functions of same argument but different order. The Hankel functions do not exhibit zeros. Since Hankel functions of different orders are linearly independent it can be concluded that the zeros in the coefficients $\breve{G}_\nu(\omega)$ are exclusively dependent on the loudspeaker directivity coefficients $\breve{G}'_\mu(\omega)$. In cases where $\breve{G}_\nu(\omega)$ becomes zeros or is small so that numerical instabilities arise, (preferably frequency dependent) regularization such as in [1] can be applied in order to yield a realizable solution. Contrary to conventional multichannel regularization, the presented approach allows for independent regularization of each mode ν of the compensation filter. Thereby, stable modes need not be regularized while the regularization of individual unstable modes can be assumed to be favorable compared to conventional regularization of the entire filter. However, note that regularization reduces the accuracy of the compensation filter.

5. RESULTS

In order to illustrate the general properties of the presented approach we consider in the following a circular distribution of highly directional secondary sources whose spatio-temporal transfer function is given by

$$\breve{G}'_\mu(\omega) = \begin{cases} \frac{M!^2}{(M+\mu)!(M-\mu)!} & \text{for } -M \le \mu \le M \\ 0 & \text{elsewhere .} \end{cases} \quad (14)$$

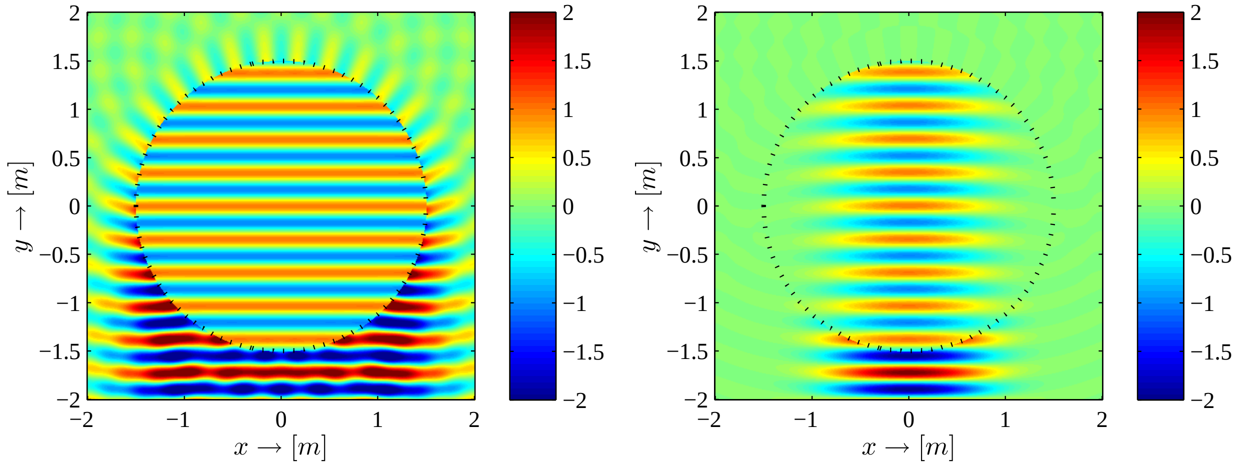
with $M = 13$. $\breve{G}'_\mu(\omega)$ given by (14) leads to a stable and defined driving function as described in section 4.
Refer to figure 2(a) for an illustration of the wave field emitted by a secondary source with a spatio-temporal transfer function given by (14).
Figure 3(a) depicts a continuous circular distribution of secondary sources with a directivity given by (14) for M reproducing a plane wave of $f_{\text{pw}} = 1000$ Hz. Consider then 3(b). It depicts a continuous circular distribution of secondary sources with a directivity given by (14) for M reproducing a plane wave of $f_{\text{pw}} = 1000$ Hz but using the conventional driving function which assumes that the secondary sources are omnidirectional. It can be seen that the wave fronts are still perfectly plane in the latter case since the timing of the driving function is appropriate. However, the energy of the reproduced virtual plane wave concentrates around the center of the secondary source distribution. In this location the secondary source transfer function that the driving function assumes is closest to the actual transfer function that the employed secondary sources exhibit.
Typically, conventional loudspeakers with closed cabinets are employed for sound field reproduction approaches like the presented one. This type of loudspeakers is close to omnidirectional for low frequencies and becomes directional for high frequencies (refer e.g. to [14]). When a distribution of such loudspeakers is driven with a driving function which assumes omnidirectional loudspeakers, the deviations of the reproduced wave field from the desired one become more pronounced for higher frequencies. This means that for off-center receiver position, an attenuation of high frequencies has to be expected.
The above described findings have been derived for a continuous distribution of secondary sources. Real-world implementation of sound field reproduction systems always employ a finite number of discrete loudspeakers. This circumstance can lead to spatial discretization artifacts. These artifacts



(a) Real part of the wave field when driven with a monochromatic signal of $f = 1000$ Hz.

(b) Polar plot of normalized secondary source far-field directivity.

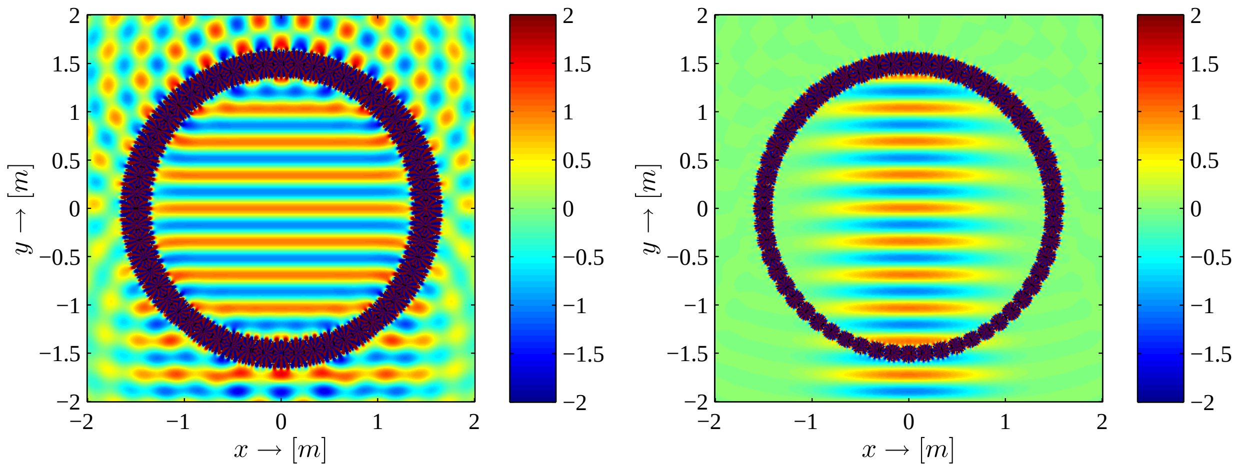
Fig. 2: Illustration of the properties of the directional secondary sources under consideration. The transfer function given by (14) with $M = 13$.



(a) Proposed driving function.

(b) Conventional driving function.

Fig. 3: Real part of the wave field reproduced by a continuous distribution of secondary sources with a transfer function given by (14). The desired wave field is a plane wave of frequency $f_{\mathrm{pw}} = 1000$ Hz. Expansion orders are limited to the interval $[-27\ 27]$.



(a) Proposed driving function.

(b) Conventional driving function.

Fig. 4: Real part of the wave field reproduced by a distribution of 56 discrete secondary sources with a transfer function given by (14) resp. depicted in figure 2(a). The desired wave field is a plane wave of frequency $f_{\mathrm{pw}} = 1000$ Hz. Expansion orders are limited to the interval $[-27\ 27]$.

have only been investigated in the literature in conjunction with omnidirectional secondary sources, e.g. [7, 8, 16]. The results hold qualitatively also for directional secondary sources but a detailed analysis is not available.

Refer to figure 4(a) in order to get a first impression of the properties of a discrete distribution of secondary sources exhibiting the above described directivity. The same array driven with the conventional driving function which assumes the secondary sources to be omnidirectional is depicted in figure 4(b).

6. CONCLUSIONS

An approach for sound field reproduction employing circular arrangements of secondary sources was presented. It was focused on the general properties of the resulting secondary source driving function when non-omnidirectional secondary sources are used. In order that the presented approach is applicable the spatio-temporal characteristics of the employed secondary sources have to be invariant with respect to rotation around the center of the secondary source arrangement. In other words, all secondary sources have to exhibit equal radiation characteristics and have to be orientated towards the center of the secondary source arrangement. It is then sufficient to know (or measure) the free-field (anechoic) radiation characteristics of only one secondary source of the distribution.

The presented approach exhibits two major benefits: (1) only a simple measurement of the secondary source distribution is required, and (2) the continuous formulation gives better insights into the fundamental properties of solution than MIMO approaches. This enables e.g. an investigation of the effects of spatial sampling which occurs in the MIMO approaches. Note that in the spatially discrete case, the presented approach is equivalent to the MIMO approaches under the assumption of rotation invariance of the secondary source transfer function.

Preliminary measurements of the ELAC 301 loudspeakers which are employed in the loudspeaker system installed at Deutsche Telekom Laboratories indicate that only very little variation in the spatio-temporal characteristics are apparent within different loudspeakers of this model. This indicates that the presented approach is indeed applicable when all loudspeakers are of the same model. However, the investigation of resulting errors when such variation in the spatio-temporal characteristics of the secondary sources is apparent or when secondary sources are not properly positioned and orientated could not be included in the present paper. Note however, that the spatio-temporal transfer function of the ELAC 301 loudspeakers is three-dimensional and requires a $2^1/_2$-dimensional respectively three-dimensional scenario.

We assumed that these directivity coefficients are precisely known. This requires high resolution measurements of the coefficients in order to assure that no considerable spatial aliasing occurs. It also advisable that the radius of the microphone array is not too different from the radius of the secondary source contour under consideration. The presented approach implicitly includes an extrapolation of the microphone array measurements to the radius of the secondary source contour. The restrictions of extrapolation of such spatially discrete data is not known. Due to the fact that each spatial mode of the compensation filter can be pre-computed offline, it is likely that the precision requirements can be met. The extension of the presented approach to $2^1/_2$-dimensional and three-dimensional reproduction has been submitted [17, 18].

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