On Beamforming and Generalized Radon Transforms in Sound Field Analysis

Till Rettberg and Sascha Spors

Institut für Nachrichtentechnik, Universität Rostock,,

18191 Rostock, Deutschland,

Email: {till.rettberg, sascha.spors}@uni-rostock.de

Introduction

For sound field analysis (SFA) with compact microphone arrays it is often desirable to obtain a signal representation in terms of incoming plane waves. Among other methods, the plane wave decomposition can be achieved by scanning the sound field with an (theoretical) ideal beamformer for all possible directions, possibly by spatial Fourier transform methods. The Radon transform (RT) [8] is another tool widely used for this purpose, notably in [1, 9, 4, 3]. Outside of acoustics, application of the RT has long history in geophysics and medical tomography. In image processing a closely related (if not identical) variant is known as the Hough transform [5], which is widely used for feature detection.
The two different plane wave decomposition methods are closely related: It has often been noted that the RT corresponds to a full-band delay-and-sum beamformer [6, 9]. This paper aims to extend this relationship to a more general RT formulation and to revisit previous transform variants in a more unified framework.

A Generalized Radon Transform

Application of the RT for wave field analysis is schematically depicted in fig.1: A microphone array is situated somewhere in space. For presentational convenience, only the two-dimensional case is regarded and discretization is neglected. Thus a (piecewise) continuous sensor in the (x, y)-plane is considered. Its contour may be described by the parametric equation $\xi \to \mathbf{x}$, where ξ is termed *offset parameter* in seismic literature. The received signal along the sensor over time is denoted by $s(\xi, t)$ in the *data space* domain, cf. fig.1. Then the Radon transform $\mathcal{R}\{s(\xi, t)\}$ maps to a signal representation $S(\mathbf{p}, \tau)$ in *model space*, with ray parameter vector $\mathbf{p}$ and intercept time τ, given by

$$S(\mathbf{p}, \tau) = \mathcal{R}\{s(\xi, t)\} = \int w(\mathbf{p}, \xi)\, s(\xi, t)\, \delta(t - f(\mathbf{p}, \xi) - \tau)\, \mathrm{d}\xi \qquad (1)$$

for a weighting function $w(\cdot)$ along the contour of integration described by the function $f(\cdot)$. Using the delta function's sifting property, this can be written alternatively as

$$S(\mathbf{p}, \tau) = \int w(\mathbf{p}, \xi)\, s(\xi, f(\mathbf{p}, \xi) + \tau)\, \mathrm{d}\xi\,. \qquad (2)$$

With a scalar ray parameter $\mathbf{p} = p$ for slope, $f(\mathbf{p}, \xi) = p\xi$

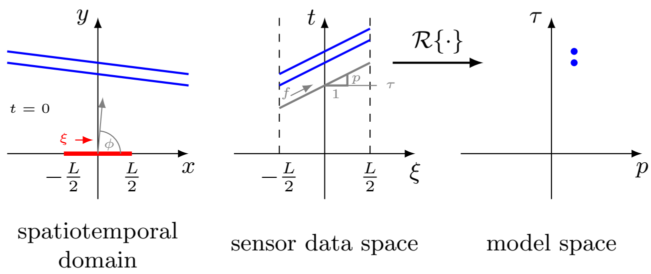

Figure 1: SFA using the Radon transform. Left: An exemplary wave field at a single time instant with e incoming wave fronts of two broad-band pulses. A linear sensor of length L positioned on the x-axis, the steering angle ϕ is indicated. Center: data space signal $s(\xi, t)$ with slope parameter p. Right: resulting model space signal $S(\mathbf{p}, \tau)$ (only maxima indicated).

and uniform weighting $w(p, \xi) = 1$, (2) reduces to the classical linear RT. As already noted, this is equivalent to delay-and-sum beamforming, with a linear array, omnidirectional sensors under far-field assumptions. For an array located on the x-axis, the slope parameter p is related to the steering angle ϕ (see fig. 1) by

$$p = -\frac{1}{c}\cos\phi\,, \qquad (3)$$

where c denotes the speed of sound.

The chosen formulation imposes two deliberate restrictions on the transform variants that can be expressed in this framework:

1. $f(\cdot)$ and $w(\cdot)$ are independent of the intercept time τ. In consequence $\mathcal{R}$ is time-invariant and its temporal Fourier-Transform $\mathcal{F}_\tau\{S(\mathbf{p}, \tau)\}$ can be interpreted as a beam pattern.
2. The curve integral in (2) is performed along (ξ, t), where t is a function of ξ. Thus $\mathcal{R}$ is distortion-free (non-convolutive) at each point ξ. Together with frequency-independent weighting, this allows for the geometric interpretation in time domain as matched filter for wave fronts.

Naturally, dropping these two limitations yields a far broader class of transforms. Other authors, e.g. Beylkin in [2], have done so for the discrete RT. However for the purpose of this paper, easier interpretation is favored over more powerful approaches.

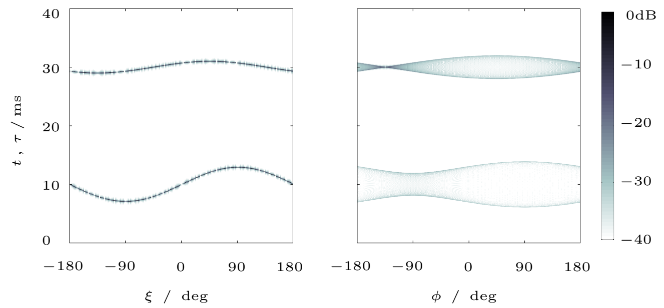

Figure 2: A circular array focussing to elevated sources. A plane wave at $\tau = 10$ms incident from $\theta = 0°$, another at $\tau = 30$ms from $\theta = 70°$. Array radius is 1m. Left: data space signal $s(\xi, t)$ with offset $\xi \in [0, 2\pi)$ Right: slice through model space signal $S(\mathbf{p}, t)$ with focussing angle $\theta = 70°$

A further remark: For arbitrary sensor geometries of finite extent, existence of a unique inverse RT cannot generally be expected. It is however not of primary interest here. $S(\mathbf{p}, \tau)$ is only considered an approximate plane wave represenation that may be subject to peak detection or other further processing steps. Similar to the treatment of the Hough Transform in image processing, the focus is rather on the forward transform and its relation to matched filtering.

Circular Arrays and Elevated Sources

While (2) can be formulated for arbitrary sensor geometries, in practice regular setups are usually employed. In twodimensional SFA, circular arrays have been a common choice due to rotational symmetry: The shape of the model space response is then invariant up to the cyclic shift. Any array of two-dimensional extent is capable to discern elevated waves, although with ambiguity of the upper and lower half space. For circular arrays this was done in [4]. Using a two-dimensional ray parameter $\mathbf{p} = (\phi, \theta)^T$ with steering angle $\phi \in [0, 2\pi)$ in azimuth and ambiguous elevation angle $\theta \in [0, \frac{\pi}{2}]$, measured from the xy-plane to the z-axis. The RT is then given by (2) with

$$f(\phi, \theta, \xi) = \frac{r_0}{c} \cos(\theta) \cos(\phi - \xi) \ , \tag{4}$$

where r_0 is the array radius. Fig.2 depicts a slice through $S(\mathbf{p}, \tau)$ for a fixed θ, that corresponds to the elevation of one of the incident wave fronts.

Non-Uniform Array Response

So far, plane waves are considered as the underlying source model and each point ξ along the sensor as an omnidirectional receiver. As a consequence of these two assumptions, the data space response of a single wave front exhibits no amplitude variation along $f(\cdot)$. Now the case is considered, when these conditions are not met.

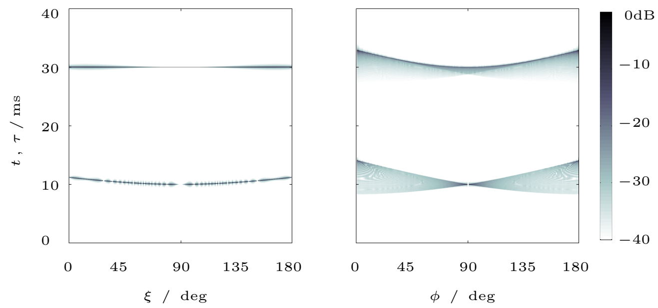

Figure 3: Focussing to near-field sources: two spherical waves incident at broadside on a linear array of 2m length. The wave front curvatures correspond to source distances $d = 1$m and $d = 100$m. Left: data space signal $s(\xi, t)$, Right: slice through $S(\mathbf{p}, \tau, d)$ for fixed $d = 1$, i.e. focussed the to near point source.

Focussing to Near-field Sources

Instead of plane waves, a model consisting of spherical waves can be adopted, by taking the wave curvature of a nearfield source into account. In [4] this was done for modal beamforming with a circular array, the delay-and-sum formulation can be found e.g. in [6].
For a focussing distance d the integration contour for a linear sensor results in

$$f(\phi, d, \xi) = \frac{1}{c}\sqrt{(\xi - d\cos\phi)^2 + (d\sin\phi)^2} \, , \tag{5}$$

and likewise for circular sensor of radius r_0 with $\Phi := \xi - \phi$ in

$$f(\phi, d, \xi) = \frac{1}{c}\left(\sqrt{(d - r_0\cos\Phi)^2 + r_0^2\sin^2\Phi} - d\right) \, . \tag{6}$$

An example of the linear case is shown in fig.3, again as a slice through $S(\mathbf{p}, \tau)$ for a fixed focussing distance d.

Nearfield beamforming raises the question of an appropriate weighting function. In contrast to the plane wave case, the response of a single source is not uniform in data space, but exhibits a decay proportional to $1/d$ associated with a spherical monopole. In dataspace, this is most prominent for a linear array and a near source in endfire direction. To account for the varying signal-to-noise ratio at the sensor input, a weighting $w(\mathbf{p}, \xi)$ proportional to the wave response can be introduced. Then the weighted RT corresponds a beamformer for spherical waves with maximum robustness in terms of white noise gain.

Higher-Order Sensors

Amplitude variations in data space also arise when non-uniform sensors are used, e.g. microphones with figure-of-eight or cardioid characteristics. Consider for instance a circular array with 1st-order directive transducers oriented in outward radial direction. For an incident plane

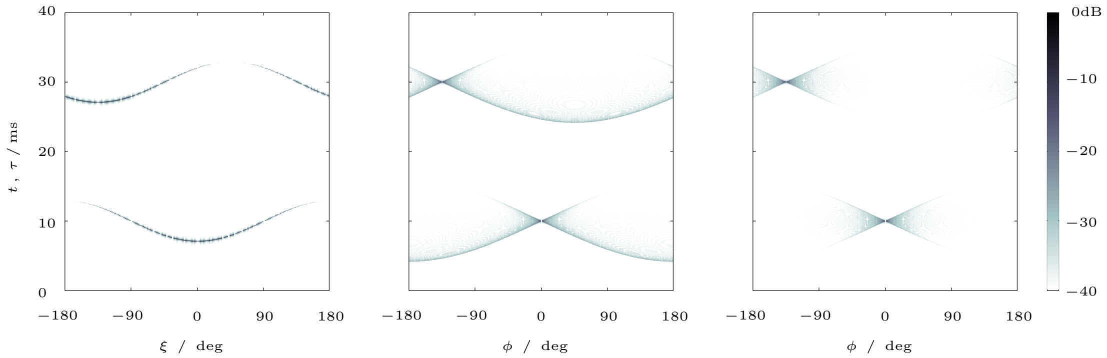

Figure 4: Weighted RT with directional sensors: Two plane waves incident on a circular array with radius $r = 1$m. Cardioid microphones oriented in outward radial direction. Left: data space signal $s(\xi, t)$, Center: $S(\mathbf{p}, \tau)$ for uniform weighting $w = 1$, Right: $S(\mathbf{p}, \tau)$ with weights $w(\phi, \xi) = (1 + \cos(\phi - \xi))/2$ corresponding to cardioid characteristic

wave from ϕ and fixed $\theta = 0°$ the response in data space along the graph of $f(\phi, \xi)$ as in (4) is given by

$$p(\xi, \phi) = \beta + (1 - \beta)\cos(\xi - \phi)\,, \tag{7}$$

with $\beta \in [0, 1]$, and specifically $\beta = 0.5$ for ideal cardioid microphones. To maintain the interpretation of a matched filter, the weights may be chosen accordingly with $w(\xi, \phi) = p(\xi, \phi)$. Fig. 4 depicts the Radon Transform with uniform weighting (center) and with the matched cardioid weighting as given by (7) (right). In this case, the weighting has also noticeable impact on the shape of the model space response: Contra-lateral contributions are attenuated and completely rejected in $-\phi$-direction, resulting in a better localized response.
In the context of SFA, Radon transforms for directive arrays have been previously considered in [3]. In order to allow wave field reconstruction, the discrete RT is extended by a DFT is performed along $f(\cdot)$. If exact reconstruction is not necessary, this approach can be simplified to the weighting given by (7), due to the fact that it represents a linear combinations of three spatial Fourier contributions. The improved localization properties are maintained.

Beside the matched filter approach, any other choices are obviously possible. For a linear array, farfield beampattern synthesis by applying suitable windowing functions is common in antenna design, e.g. [10, chapter 3]. The employed framework is also equivalent to generalized cross correlation methods used for direction-of-arrival estimation [7].

Conclusion

To obtain an approximate plane wave decomposition, both Radon transform and beamforming are appropriate tools in SFA. The generalized RT as utilized here and weighted delay-and-sum beamforming can be regarded as equivalent frameworks. For the case of nearfield beamforming and/or non-omnidirectional sensors, weighting proportional the amplitude response of the desired event, generalizes the uniform delay-and-sum beamformer to matched filtering for maximum robustness.

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