On the Connections of Wave Field Synthesis and Spectral Division Method Plane Wave Driving Functions

Frank Schultz and Sascha Spors
Institute of Communications Engineering, University of Rostock, R.-Wagner-Str. 31 (Haus 8), D-18119 Rostock, Germany
E-Mail: {frank.schultz,sascha.spors}@uni-rostock.de

Introduction

Wave Field Synthesis (WFS) [1–3] is a well-established sound field synthesis (SFS) technique that uses a dense spatial distribution of loudspeakers arranged around an extended listening area. It has been shown that WFS based on the Neumann Rayleigh integral constitutes the high-frequency and/or farfield approximation of the explicit SFS solution, such as the Spectral Division Method (SDM) [4] and Nearfield Compensated Higher-Order Ambisonics (NFC-HOA) [5,6]. However, for SFS of a virtual plane wave using a linear loudspeaker array a mismatch between an SDM and a WFS driving function has been reported in [7].

In this paper we will derive WFS plane wave driving functions using a similar stationary phase approximation as introduced for the virtual non-focused point source, cf. [2]. This yields WFS driving functions either for a reference point or for a parallel reference line. It is shown that the latter is identical to the high-frequency and/or farfield approximated SDM solution.

Neumann Rayleigh Integral

The paper considers continuous secondary source distributions (SSDs). WFS using spherical monopoles as secondary sources is based on the Neumann Rayleigh integral as the forward wavefield propagator

$$P(\mathbf{x},\omega) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} -2 \frac{\partial P(\mathbf{x_0'},\omega)}{\partial n} G(\mathbf{x},\mathbf{x_0'},\omega)\,\mathrm{d}z_0 \right] \mathrm{d}y_0 \tag{1}$$

here using $\mathbf{x} = (x,y,z)^\mathrm{T}$, the SSD in the yz-plane denoted as $\mathbf{x_0'} = (0,y_0,z_0)^\mathrm{T}$, the unit inward normal $\mathbf{n}(\mathbf{x_0'}) = (1,0,0)^\mathrm{T}$ and the 3D freefield Green's function

$$G(\mathbf{x},\mathbf{x_0'},\omega) = \frac{1}{4\,\pi} \frac{\mathrm{e}^{-\mathrm{j}\,\frac{\omega}{c}\,|\mathbf{x}-\mathbf{x_0'}|}}{|\mathbf{x}-\mathbf{x_0'}|}. \tag{2}$$

The directional derivative is defined as

$$\frac{\partial P(\mathbf{x_0'},\omega)}{\partial n} = \langle \mathrm{grad}_{\,\mathbf{x}} P(\mathbf{x},\omega)\big|_{\mathbf{x}=\mathbf{x_0'}}\,,\,\mathbf{n}(\mathbf{x_0'})\rangle. \tag{3}$$

with the dot product notation $\langle\cdot,\cdot\rangle$. The $\mathrm{e}^{+\mathrm{j}\,\omega\,t}$ time convention for monochromatic waves is deployed. c denotes the speed of sound and ω the angular frequency. The considered listening area is the half space in direction of $\mathbf{n}$, i.e. $x > 0$ for the chosen geometry.

3D Neumann WFS of a Plane Wave

The directional derivative (3) of a plane wave with propagating direction $\mathbf{n_{PW}}$

$$P_\mathrm{PW}(\mathbf{x},\omega) = P(\omega)\,\mathrm{e}^{-\mathrm{j}\,\frac{\omega}{c}\,\langle \mathbf{n_{PW}},\mathbf{x}\rangle} \tag{4}$$

is calculated to

$$\frac{\partial P_\mathrm{PW}(\mathbf{x_0'},\omega)}{\partial n} = -\mathrm{j}\,\frac{\omega}{c}\,\cos\varphi_p'\,P_\mathrm{PW}(\mathbf{x_0'},\omega) \quad \text{with} \tag{5}$$

$$\cos\varphi_p' = \langle \mathbf{n_{PW}},\mathbf{n}(\mathbf{x_0'})\rangle \tag{6}$$

using the angle φ_p' between $\mathbf{n_{PW}}$ and $\mathbf{n}(\mathbf{x_0'})$. The 3D Neumann WFS driving function thus reads

$$D(\mathbf{x_0'},\omega) = -2\frac{\partial P(\mathbf{x_0'},\omega)}{\partial n} = 2\mathrm{j}\frac{\omega}{c}\cos\varphi_p' P_\mathrm{PW}(\mathbf{x_0'},\omega) \tag{7}$$

requiring plane wave propagating directions that fulfill $\cos\varphi_p' > 0$.

3D to 2.5D Neumann WFS

2.5D SFS using a linear SSD built from spherical monopoles is given by the synthesis equation [7, (9)]

$$P(\mathbf{x},\omega) = \int_{-\infty}^{+\infty} D(\mathbf{x_0},\omega)\,G(\mathbf{x},\mathbf{x_0},\omega)\,\mathrm{d}y_0 \tag{8}$$

using the specific geometry $\mathbf{x_0} = (0,y_0,0)^\mathrm{T}$, the SSD on the y-axis $\mathbf{x} = (x,y,0)^\mathrm{T}$ and $\mathbf{n}(\mathbf{x_0}) = (1,0,0)^\mathrm{T}$. The plane wave propagating direction is restricted to the xy-plane as

$$\mathbf{n_{PW}} = (\cos\varphi_\mathrm{PW},\sin\varphi_\mathrm{PW},0)^\mathrm{T} \quad \text{with} \tag{9}$$

$$\cos\varphi_p = \langle \mathbf{n_{PW}},\mathbf{n}(\mathbf{x_0})\rangle > 0. \tag{10}$$

The implicit derivation of the plane wave driving function to be used with (8) is similar to that of a virtual non-focused point source, cf. [2, Ch. 3.1]. This is shown as follows.

Stationary Phase Approximation for a Reference Point

For the virtual plane wave the inner integral of (1) reads, cf. [8, (5.4-5.6)]

$$I = \int_{-\infty}^{+\infty} P(\omega)\,2\,\mathrm{j}\,\frac{\omega}{c}\,\cos\varphi_\mathrm{PW}\,\mathrm{e}^{-\mathrm{j}\frac{\omega}{c}\sin\varphi_\mathrm{PW}\cdot y_0}\,G(\mathbf{x},\mathbf{x_0'},\omega)\,\mathrm{d}z_0 \tag{11}$$

using $\mathbf{x} = (x, y, 0)^{\mathrm{T}}$. With the stationary phase approximation (SPA) [2, (3.2-3.4)], [9, Ch. 4.6.1], integrals of the kind

$$I = \int_{-\infty}^{+\infty} f(z_0)\,\mathrm{e}^{+\mathrm{j}\,\phi(z_0)}\,\mathrm{d}z_0 \tag{12}$$

can be approximated as

$$I \approx \sqrt{\frac{2\,\pi}{|\phi''(z_{0,s})|}}\, f(z_{0,s})\,\mathrm{e}^{+\mathrm{j}\,\phi(z_{0,s})}\,\mathrm{e}^{+\mathrm{j}\,\frac{\pi}{4}\mathrm{sign}[\phi''(z_{0,s})]} \tag{13}$$

for $\phi(z_0) \to \infty$. Note the different use of ϕ and φ. $z_{0,s}$ is the stationary phase point at which $\phi'(z_{0,s}) = 0$ and $\phi''(z_{0,s}) \neq 0$ hold (note the derivatives w.r.t. z_0). Eq. (11) can be brought into the form of (12) and for

$$|-\frac{\omega}{c}|\mathbf{x} - \mathbf{x_0'}| - \frac{\omega}{c}\sin\varphi_{\mathrm{PW}}\, y_0| \gg 1 \tag{14}$$

the stationary phase point $z_{0,s} = 0$ can be derived. Then the approximation (13) yields the integrand of (8)

$$I \approx D(y_0, \omega)\, G(\mathbf{x}, \mathbf{x_0}, \omega) \tag{15}$$

and the driving function is given as

$$D(y_0, \omega) = P(\omega)\sqrt{8\pi\mathrm{j}\,\frac{\omega}{c}|\mathbf{x} - \mathbf{x_0}|}\,\cos\varphi_{\mathrm{PW}}\,\mathrm{e}^{-\mathrm{j}\,\frac{\omega}{c}\,\sin\varphi_{\mathrm{PW}}\cdot y_0}. \tag{16}$$

This driving function depends on the receiver position $\mathbf{x}$, for which typically a reference position $\mathbf{x_{Ref}}$ is chosen. In [10] it is shown that correct synthesis is not only achieved at exactly $\mathbf{x_{Ref}}$ but rather along a parametric curve, which is analytically given for different referencing schemes in [10], such as the here discussed reference point and parallel reference line.

For an arbitrarily located linear SSD along $\mathbf{x_0}$ equation (16) is given as

$$\begin{aligned} D_{\mathrm{RefPoint}}(\mathbf{x_0}, \omega) =& P_{\mathrm{PW}}(\mathbf{x_0}, \omega)\sqrt{\mathrm{j}\,\frac{\omega}{c}} \times \\ & \sqrt{8\,\pi\,|\mathbf{x_{Ref}} - \mathbf{x_0}|}\,\langle \mathbf{n_{PW}}, \mathbf{n}(\mathbf{x_0})\rangle \end{aligned} \tag{17}$$

requiring coplanarity of $\mathbf{x} - \mathbf{x_0}$, $\mathbf{x_{Ref}} - \mathbf{x_0}$, $\mathbf{n}(\mathbf{x_0})$, $\mathbf{n_{PW}}$ and $\cos\varphi_p > 0$.

This driving function type (SPA I) is identical to the virtual point source's type in [2, (3.10&3.11)].

Stationary Phase Approximation for a Reference Line

Applying a subsequent, second SPA w.r.t. y_0 on the integral (8) with the driving function (16) for finding the stationary phase point $y_{0,s}$ at which

$$\phi'(y_0 = y_{0,s}) = 0 \tag{18}$$

holds, yields the condition [11], cf. [10]

$$(y - y_{0,s})^2 = x^2\,\tan^2\varphi_{\mathrm{PW}} \tag{19}$$

when

$$|-\frac{\omega}{c}(|\mathbf{x} - \mathbf{x_0}| + \sin\varphi_{\mathrm{PW}}\, y_0)| \gg 1 \tag{20}$$

holds. Plugging the solution for the stationary phase point $y_0 = y_{0,s}$ into $|\mathbf{x} - \mathbf{x_0}|$ of (16) yields

$$|\mathbf{x} - \mathbf{x_0}| = x\sqrt{1 + \tan^2\varphi_{\mathrm{PW}}}. \tag{21}$$

Inserting (21) into (16) by considering the allowed plane wave directions $-\frac{\pi}{2} < \varphi_{\mathrm{PW}} < +\frac{\pi}{2}$ yields the driving function

$$D(y_0, \omega) = P(\omega)\sqrt{8\,\pi\,\mathrm{j}\,\frac{\omega}{c}\,\cos\varphi_{\mathrm{PW}}\, x}\cdot\mathrm{e}^{-\mathrm{j}\,\frac{\omega}{c}\,\sin\varphi_{\mathrm{PW}}\cdot y_0} \tag{22}$$

for the synthesis integral (8).

This driving function is dependent on x, i.e. holds for a line parallel to the linear SSD. Typically a reference distance x_{Ref} is chosen at which SFS is correct in amplitude and phase.

For an arbitrarily located linear SSD along $\mathbf{x_0}$ and the distance r_{Ref} between the SSD and the parallel reference line, (22) reads

$$\begin{aligned} D_{\mathrm{RefLine}}(\mathbf{x_0}, \omega) =& P_{\mathrm{PW}}(\mathbf{x_0}, \omega)\sqrt{\mathrm{j}\,\frac{\omega}{c}} \times \\ & \sqrt{8\,\pi\, r_{\mathrm{Ref}}}\,\sqrt{\langle \mathbf{n_{PW}}, \mathbf{n}(\mathbf{x_0})\rangle} \end{aligned} \tag{23}$$

requiring coplanarity of $\mathbf{x} - \mathbf{x_0}$, $\mathbf{n}(\mathbf{x_0})$ and $\mathbf{n_{PW}}$ as well as $\cos\varphi_p > 0$.

This driving function type (SPA II) is identical to the virtual point source's type in [2, (3.16&3.17)].

Eq. (23) is precisely the same result as [7, (29)], i.e. the farfield/high-frequency approximation of the exact 2.5D SDM solution [7, (17)], in detail shown in [11, Ch. 2] and [10].

Comparison of SPA I and II

The reported mismatch [7, Sec. IV-B], [6, Ch. 3.9.4] between the 2.5D SDM and 2.5D WFS solution stems from the invalid assumption that a 'moving' receiver/reference point

$$|\mathbf{x} - \mathbf{x_0}| = |\begin{pmatrix} x \\ y_0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ y_0 \\ 0 \end{pmatrix}| = x \tag{24}$$

in the SPA I solution (16) yields the correct approximation for referencing to a parallel line [6, Ch. 3.9.3]. Then the comparison of the two solutions

- (16) as $D_{\mathrm{SPAI}}(y_0, \omega)$ with $|\mathbf{x} - \mathbf{x_0}| = x$
- (22) as $D_{\mathrm{SPAII}}(y_0, \omega)$

leads to cf. [7, (28,29)]

$$D_{\mathrm{SPAI}}(y_0, \omega) = D_{\mathrm{SPAII}}(y_0, \omega)\sqrt{\cos\varphi_{\mathrm{PW}}}$$

and led to conclusion that WFS differs from the SDM solution for large φ_{PW}.

In fact, both solutions are correct – using a specific parametric curve referencing according to [10] – either for a reference point (SPA I, (17)) or for a parallel reference line (SPA II, (23)). The latter one – WFS w.r.t. a reference line – is identical to the farfield/high-frequency approximation of the explicit 2.5D SDM solution, that inherently involves a referencing to a parallel line. This also holds for the virtual point source [4, 11] as well as for moving virtual point sources [12, 13].

Conclusion

This paper outlined the stationary phase approximations deriving the 2.5D Neumann Rayleigh integral from its 3D version for Wave Field Synthesis of a virtual plane wave with a linear array. It is discussed that this is the same approach as for the virtual point source and that the derivations are consistent for the virtual plane and spherical wave. Thus, the first approximation yields driving functions for a reference point, the second one yields driving functions for a parallel reference line. The latter are precisely identical with the farfield/high-frequency approximated explicit Spectral Division Method solutions.

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References

[1] Berkhout, A.J. (1988): "A holographic approach to acoustic control." In: *J. Audio Eng. Soc.*, **36**(12):977–995.

[2] Start, E.W. (1997): *Direct Sound Enhancement by Wave Field Synthesis*. Ph.D. thesis, Delft University of Technology.

[3] Spors, S.; Rabenstein, R.; Ahrens, J. (2008): "The theory of Wave Field Synthesis revisited." In: *Proc. of 124th Audio Eng. Soc. Convention, Amsterdam*, #7358.

[4] Spors, S.; Ahrens, J. (2010): "Analysis and improvement of pre-equalization in 2.5-dimensional Wave Field Synthesis." In: *Proc. of 128th Audio Eng. Soc. Convention, London*, #8121.

[5] Ahrens, J.; Spors, S. (2009): "Alterations of the temporal spectrum in high-resolution sound field reproduction of different spatial bandwidths." In: *Proc. of 126th Audio Eng. Soc. Convention, Munich*, #7742.

[6] Ahrens, J. (2012): *Analytic Methods of Sound Field Synthesis*. Heidelberg: Springer, 1. ed.

[7] Ahrens, J.; Spors, S. (2010): "Sound field reproduction using planar and linear arrays of loudspeakers." In: *IEEE Trans. Audio Speech Language Process.*, **18**(8):2038–2050.

[8] Spors, S. (2006): *Active Listening Room Compensation for Spatial Sound Reproduction Systems*. Ph.D. thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg.

[9] Williams, E.G. (1999): *Fourier Acoustics, Sound Radiation and Nearfield Acoustic Holography*. London, San Diego: Academic Press, 1. ed.

[10] Firtha, G.; Fiala, P.; Schultz, F.; Spors, S. (2016): "Referencing the WFS driving functions." In: *IEEE Trans. Audio Speech Language Process. (in preparation)*.

[11] Schultz, F. (2016): *Sound Field Synthesis for Line Source Array Applications in Large-Scale Sound Reinforcement*. Ph.D. thesis, University of Rostock.

[12] Firtha, G.; Fiala, P. (2015): "Sound field synthesis of uniformly moving virtual monopoles." In: *J. Audio Eng. Soc.*, **63**(1/2):46–53.

[13] Firtha, G.; Fiala, P. (2015): "Wave field synthesis of moving sources with retarded stationary phase approximation." In: *J. Audio Eng. Soc.*, **63**(12):958–965.