On the Connections of High-Frequency Approximated Ambisonics and Wave Field Synthesis
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**Introduction**

Wave Field Synthesis (WFS) constitutes a ray-based, implicit solution of the sound field synthesis (SFS) problem based on the Helmholtz integral equation \([1]\). Explicit solutions derived in the modal domain are known for simple SFS geometries and fundamental virtual source types.

In \([1]\) we have shown equivalence — using a linear array as secondary source distribution (SSD) — of 2/2-dimensional (2D) WFS and a high-frequency/far-field approximation of the so called Spectral Division Method (SDM) \([2, \text{ Sec. 3.7}]\), which constitutes the explicit SFS solution in Cartesian coordinates.

Near-field Compensated Higher Order Ambisonics (NFC-HOA) \([2, \text{ Sec. 3.5}]\) is known as the explicit SFS solution for spherical/circular SSD geometry, for which an equivalence with WFS is assumed as well. In \([2, \text{ Sec. 4.4.2}]\) it was stated that WFS constitutes a high-frequency approximation of Near-field Compensated Infinite Order Ambisonics by numerically evaluating the driving filter’s Fourier coefficients. In the present contribution further aspects and analytic calculus are given to reinforce this claim for 2.5D SFS.

**2.5D Sound Field Synthesis**

To compare NFC-HOA and WFS for 2.5D SFS, a circular SSD is required as the modal expansion of HOA is based on this geometry. The weighted superposition of monochromatic sound fields emanated by spherical monopoles reads for \(e^{j\omega t}\) time convention

\[
P(x, \omega) = \int_{0}^{2\pi} D(x_0, \omega) \frac{e^{-j\frac{2\pi}{\lambda}|x-x_0|}}{4\pi|x-x_0|} r_0 \, d\phi_0,
\]

with speed of sound \(c\), angular frequency \(\omega\), imaginary unit \(j\) and SSD radius \(r_0\). Secondary sources are located at \(x_0 = (r_0 \cos \phi_0, r_0 \sin \phi_0)^T\). Listening positions \(r < r_0\) are denoted by \(x = (r \cos \phi, r \sin \phi)^T\). Vector magnitude is denoted as \(|\cdot|\). Inner vector product is written as \((\cdot, \cdot)\).

The inward unit normal of the SSD contour at position \(x_0\) is given as \(\hat{n}_0(x_0) = -\frac{x_0}{r_0}\). For the discussion we use dimensionless \(k r_0 = \frac{2\pi}{\lambda} r_0\) and \(\frac{\lambda}{\lambda}\) with the wave length \(\lambda\).

Different driving filters \(D_{\text{Ref},x}(x, \omega)\) realize sound fields of virtual source types \(S(x, \omega)\), such as e.g. i) a point source \(S_{\text{PS}}(x, \omega) = e^{-j\frac{2\pi}{\lambda}|x-x_\text{PS}|}\) at position \(x_\text{PS} = (r_{\text{PS}} \cos \phi_{\text{PS}}, r_{\text{PS}} \sin \phi_{\text{PS}})^T\) with \(r_{\text{PS}} > r_0\) and ii) a plane wave \(S_{\text{PW}}(x, \omega) = e^{-j\frac{2\pi}{\lambda}(k_{\text{PW}}, \hat{x})}\) propagating into direction of the unit vector \(k_{\text{PW}} = (\cos \phi_{\text{PW}}, \sin \phi_{\text{PW}})^T\).

**WFS Driving Filters**

The WFS driving filter for a virtual point source reads \([3, (2.137)]\)

\[
D_{\text{PS}, \text{WFS}}(x_0, \omega) = w_{\text{PS}}(x_0) \sqrt{\frac{\lambda}{c}} e^{\frac{j\lambda}{4\pi}|x_0-x_{\text{PS}}|} \left(\sin \frac{|x_0-x_{\text{PS}}|}{\pi r_{\text{PS}}} \right)
\]

with the local wavenumber vector \([4, 1]\) \(k_{\text{PS}}(x_0) = \frac{x_{\text{PS}}-x_0}{|x_{\text{PS}}-x_0|}\) and the spatial secondary source window \(w_{\text{PS}}(x_0) = 1\) if \(\langle k_{\text{PS}}(x_0), \hat{n}_0(x_0) \rangle > 0\) , zero otherwise. The WFS driving filter for a virtual plane wave reads \([3, (2.177)]\)

\[
D_{\text{PW}, \text{WFS}}(x_0, \omega) = w_{\text{PW}}(x_0) \sqrt{\frac{\lambda}{c}} e^{\frac{j\lambda}{4\pi}|x_0-x_{\text{PW}}|} \left(\sin \frac{|x_0-x_{\text{PW}}|}{\pi r_{\text{PW}}} \right)
\]

with the spatial secondary source window \(w_{\text{PW}}(x_0) = 1\) if \(\langle k_{\text{PW}}(x_0), \hat{n}_0(x_0) \rangle > 0\) , zero otherwise. Positions of amplitude correct SFS are defineable by the specific referencing scheme linked to \(x_{\text{Ref}}(x_0)\), cf. the referencing function discussed in \([4]\).

**NFC-HOA Driving Filters**

The NFC-HOA driving filter for a virtual point source reads \([2, \text{ Ch. 5}]\)

\[
D_{\text{PS}, \text{HOA}}(x_0, \omega) = \frac{1}{2\pi r_0} \sum_{m=-M}^{+M} \frac{h_{2m}(\frac{2\pi}{\lambda} x_{\text{PS}})}{\sqrt{|m|}} e^{j m (\phi_0 - \phi_{\text{Ref}})}
\]

using the spherical Hankel function \(h_{2m}(\cdot)\) of second kind and order \(\nu\) \([5, \text{ Ch. 10}]\). The NFC-HOA driving function for a virtual plane wave reads \([2, \text{ Ch. 5}]\)

\[
D_{\text{PW}, \text{HOA}}(x_0, \omega) = \frac{2j}{c r_0} \sum_{m=-M}^{+M} \frac{(-j)^{|m|}}{\sqrt{|m|}} e^{j m (\phi_0 - \phi_{\text{PW}})}
\]

Typically, modal order \(M \leq \frac{L}{2} - 1\) for a discretized SSD of even \(L\) secondary sources is utilized to avoid spatial aliasing \([2, \text{ Sec. 4.4.1}]\). In this paper, we are rather interested for the cases of large arguments in Hankel functions and \(M \to \infty\), i.e. a far-field/high-frequency approximation and infinite modal (i.e. spatial) bandwidth.
The SPA of (1) using $D_{PS\cdot WFS}(x_0,\omega)$ and $D_{PW\cdot WFS}(x_0,\omega)$ precisely results in the correct amplitude and phase of the intended virtual sound fields $S_{PS}(x,\omega)$ and $S_{PW}(x,\omega)$, respectively, cf. [7]. This is due to the single stationary secondary source $x_0 = (r_0, \phi_0)$ that is found along the line $x_{PS} \to x_0 \to 0$ for a virtual point source and along the line $K_{PW} \to x_0 \to 0$ (where active secondary source selection holds as well) for a virtual plane wave, respectively.

The SPA of (1) using $D_{PS\cdot HOA}(x_0,\omega)$ and $D_{PW\cdot HOA}(x_0,\omega)$ in general follows the same principles, cf. [7]. Since the SPA relies on far-field/high-frequency assumptions, it is meaningful to apply a large argument approximation $h_{0j}(z) \approx j^{\nu + 1} e^{\frac{jz}{r}}$ within the driving functions first (which by itself is an SPA, the derivation can thus be performed in different ways). For the virtual point source the large argument approximation $\frac{e}{c} r_{PS} \to \infty$, $\frac{e}{c} r_{PW} \to \infty$ yields

$$D_{PS\cdot HOA}(x_0,\omega) \approx \frac{1}{2\pi r_0} \frac{e^{-j\frac{\pi}{2} r_{PW}}}{e^{-j\frac{\pi}{2} r_{PS}}} \frac{r_0}{r_{PS}} \sum_{m=-M}^{M} e^{im(\phi_0 - \phi_{PW})}$$

(6)

and for $M \to \infty$ the Dirichlet kernel evolves to the Dirac delta function [6, Sec. 1.15], resulting in

$$D_{PS\cdot HOA}(x_0,\omega) \approx \frac{1}{r_0} \frac{e^{-j\frac{\pi}{2} r_{PW}}}{e^{-j\frac{\pi}{2} r_{PS}}} \delta(\phi_0 - \phi_{PS}).$$

(7)

For the virtual plane wave the large argument approximation $\frac{e}{c} r_{PS} \to \infty$ yields

$$D_{PW\cdot HOA}(x_0,\omega) \approx \frac{2}{r_0} \frac{r_0}{e^{-j\frac{\pi}{2} r_{PS}}} \sum_{m=-M}^{M} e^{im(\phi_0 - \phi_{PW})}$$

(8)

with the angle $\phi_{PW} = \phi_{PW} - \pi$ for plane wave incidence rather than propagating direction $\phi_{PW}$. For $M \to \infty$ follows

$$D_{PW\cdot HOA}(x_0,\omega) \approx \frac{1}{r_0} \frac{4\pi r_0}{e^{-j\frac{\pi}{2} r_{PW}}} \delta(\phi_0 - \phi_{PW}).$$

(9)

This case is most illustrative: By inserting driving filter (9) into (1), the sound field is generated only by the secondary source $x_0$ where $\phi_0 = \phi_{PW}$. The complex weight $4\pi r_0 e^{j\frac{\pi}{2} r_{PW}}$ of this stationary secondary source compensates its amplitude decay towards the plane wave’s unit gain amplitude in the origin and compensates its phase shift towards the intended zero-phase of the plane wave in the origin. These SPA treatments indicate that WFS and Near-field Compensated Infinite Order Ambisonics yield identical results at the origin.

**Fourier Series**

In further search of similarity or even identity of WFS and NFC-IOA not only for some positions or a single location, but generally for the synthesized sound field, the equivalence of driving functions either in spatial or Fourier series domain

$$D_{WFS}(x_0,\omega) \approx D_{NFC\cdot \infty OA}(x_0,\omega)$$

$$D_{WFS}(m,\omega) \approx D_{NFC\cdot \infty OA}(m,\omega)$$
would be required. While WFS is inherently given in spatial domain, NFC-HOA is analytically given as spatial Fourier coefficients. Transferring both approaches to the respective corresponding domain is not straightforward when aiming for analytical solutions. Here, we follow [2, Ch. 4.4.2], calculating a Fourier series of the WFS driving filter. This is performed for the virtual plane wave in detail. The Fourier series analysis [5, (1.84)]

$$D_{\text{WFS}}(m, \omega) = \frac{1}{2\pi} \int_{0}^{2\pi} D_{\text{WFS}}(x_0, \omega) \cdot e^{-jm\phi_0} \, d\phi_0$$

for the virtual plane wave (3) with $x_{\text{Ref}} = 0$ leads to [7]

$$D_{\text{PW, WFS}}(m, \omega) = -\sqrt{8\pi r_0} \frac{\omega}{c}$$

\begin{align*}
\left[\frac{e^{-jm\phi_{PW}}}{2} \cdot (J_{m-1}(\frac{\omega}{c} r_0) - J_{m+1}(\frac{\omega}{c} r_0)) \right] *_{m} \\
\left[\frac{1}{2\pi} m (e^{-jm(\phi_{PW} + \frac{\pi}{2})} - e^{-jm(\phi_{PW} - \frac{\pi}{2})}) \right],
\end{align*}

denoting the $\nu$-th order cylindrical Bessel function of first kind by $J_{\nu}(\cdot)$ [5, Ch. 10].

The Fourier coefficient $D_{\text{PW, WFS}}(m, \omega)$ results from a convolution of the sound field specific term within the first brackets (Fig. 4 top: blue) and the secondary source selection window in the second bracket (Fig. 4 top: red). Note that the virtual source independent term $J_{m-1}(\frac{\omega}{c} r_0) - J_{m+1}(\frac{\omega}{c} r_0)$, the extended secondary source selection window $dJ_{m}(\frac{\omega}{c} r_0)$ is multiplied to the virtual source dependent term, here $e^{-jm\phi_{PW}}$ for the plane wave, cf. [8] for detailed treatment.

The analytic complex-valued convolution in the $m$-domain is subject for further research. Numerical convolution of the analytic terms and comparison against the Fourier coefficients of the NFC-HOA plane wave driving filter (5)

$$D_{\text{PW, HOA}}(m, \omega) = \frac{2j}{\pi r_0} h^{(2)}_{m}(\frac{\pi}{2} r_0) e^{-jm\phi_{PW}}$$

is explored in the following. For that we utilize the normalized spatial frequency variable $m / |kr_0|$. According to [2, Sec. 2.2] the region $|m / |kr_0|| > 1$ indicates evanescent waves, whereas $|m / |kr_0|| < 1$ marks the spatial bandwidth of propagating waves.

In Fig. 3, the level of Fourier coefficients for a virtual plane wave is depicted for NFC-HOA vs. WFS for rather large $kr_0 = 200$. In the propagating wave region the levels highly match, whereas WFS exhibits more energy in the evanescent wave region. This is again due to the discontinuous secondary source selection window compared to the smoother driving control in NFC-HOA, cf. [3, p.95]. This artifact only completely vanishes when $kr_0 \to \infty$. This in turn requires infinite spatial bandwidth $M \to \infty$ for NFC-HOA.

A more detailed picture on the Fourier coefficients is presented in Fig. 4 using rather low $kr_0 = 20$ for clarity. The top row shows the real and imaginary part of the WFS-specific functions (11) involved in the convolution. The middle and bottom rows compare NFC-HOA against WFS w.r.t. real and imaginary part, magnitude and level of Fourier coefficients. These plots indicate good similarity, but no equivalence of the two approaches. Here, for WFS even higher energy for evanescent waves can be observed compared to Fig. 3 with $kr_0 = 200$.

**Conclusion**

Although a strict proof is not yet available, the present treatment supports the initial claim, that Near-field Compensated Infinite Order Ambisonics and Wave Field Synthesis with inherently infinite spatial bandwidth behave identically if i) referencing and expansion to the origin is performed and ii) high-frequency/far-field assumptions are fulfilled. The normalized wave number $kr_0 \to \infty$ then requires modal order $M \to \infty$ and vice versa.

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**References**


Figure 4: Fourier coefficients of plane wave driving filter for NFC-HOA vs. WFS, $kr_0 = 20$, $\phi_{PW} = -\frac{\pi}{4}$. **Top row:** WFS split into real (left) and imaginary (right) part, sound field specific part (blue) $*_m$ spatial secondary source selection window (red). **Middle row:** comparison of Fourier coefficients, real (left) and imaginary (right), NFC-HOA (blue) vs. WFS (red). **Bottom row:** magnitude (left) and level (right) of middle row scenario. Note that the Fourier coefficients are continuously plotted for convenience over normalized values $m / \lceil kr_0 \rceil$. Plots created with [7, nfc_hoa_vs_WFS_drivingfunctions_PW.py].