Stochastic Signals
(Noise)

Bandpass Noise

- lowpass noise

$$\underline{n}(t) = x(t) + j\,y(t)$$

- correlation function of the stationary lowpass noise

$$\underline{R}_{nn}(t) = E\{\underline{n}^*(\tau)\;\underline{n}(\tau+t)\}$$

$$= \left(R_{xx}(t) + R_{yy}(t)\right) + j\left(R_{xy}(t) - R_{yx}(t)\right)$$

- bandpass noise

$$w(t) = \sqrt{2}\,\mathrm{Re}\left(\underline{n}(t)\,e^{j\omega_0 t}\right)$$

$$= \sqrt{2}\,x(t)\cos(\omega_0 t) - \sqrt{2}\,y(t)\sin(\omega_0 t)$$

- correlation function of the bandpass noise

$$R_{ww}(t) = E\{w(\tau)\,w(\tau+t)\}$$

$$= \left(R_{xx}(t) + R_{yy}(t)\right)\cos(\omega_0 t)$$
$$+ \left(-R_{xy}(t) + R_{yx}(t)\right)\sin(\omega_0 t)$$
$$+ \left(R_{xx}(t) - R_{yy}(t)\right)\cos(2\omega_0 t + \omega_0 \tau)$$
$$+ \left(-R_{xy}(t) - R_{yx}(t)\right)\sin(2\omega_0 t + \omega_0 \tau)$$

Remark: noise is always assumed to be zero mean, i.e., correlation and covariance are the same

Stationary Bandpass Noise

- The correlation function must not depend on τ!

- the equivalent lowpass noise must be circular symmetric (a small time shift in the bandpass domain corresponds to a rotation in the complex plane in the lowpass domain):

  $R_{xx}(t) = R_{yy}(t)$

  $R_{xy}(t) = -R_{yx}(t) = -R_{xy}(-t)$ (the last identity is a general property of cross-correlation functions)

- lowpass to bandpass transform of the correlation function

  $R_{ww}(t) = \mathrm{Re}\left(\underline{R}_{nn}(t)\, e^{j\omega_0 t}\right)$

Wiener-Khintchin Theorem

power density spectrum is the Fourier transform of the correlation function:

$$S_{ww}(\omega) = \mathcal{F}\{R_{ww}(t)\}$$

$$S_{nn}(\omega) = \mathcal{F}\{\underline{R}_{nn}(t)\}$$

(always real due to symmetry of correlation functions)

Power (Variance)

$$P = \sigma^2 = E\{w^2(t)\} = R_{ww}(0)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{ww}(\omega)\, d\omega$$

$$= E\{|\underline{n}(t)|^2\} = \underline{R}_{nn}(0)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{nn}(\omega)\, d\omega$$

White Noise

- constant power density spectrum

$$S_{nn}(\omega) = N_0$$

$\Rightarrow$ infinite power!
idealized model

- correlation function

$$R_{nn}(t) = \mathcal{F}^{-1}\{N_0\} = N_0\,\delta(t)$$

$\Rightarrow$ noise at different time instances is uncorrelated