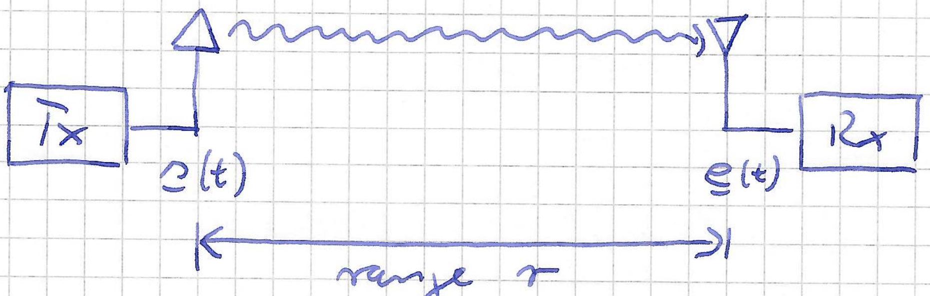
Range Estimation

One Way Channel (Radio Navigation)



- delay $\Delta t = \frac{r}{c_0}$

  ⇒ delay and range domain are the same except for a scaling by $1/c_0$

- received signal

  unknown complex factor incorporating attenuations and phase shifts

$$\underline{e}(t) = \underline{c}\,\underline{s}(t-\Delta t) + \underline{n}(t)$$

$$\circ\!\!-\!\!\bullet \qquad\qquad \circ\!\!-\!\!\bullet \qquad\qquad \circ\!\!-\!\!\bullet$$

$$\underline{E}(\omega) = \underline{c}\,\underline{S}(\omega)\,e^{-j\omega\Delta t} + \underline{N}(\omega)$$

- mathematically, range estimation is a harmonic retrieval problem (time t takes the role of the frequency)

Least Squares Delay Estimator

$$\widehat{\Delta t} = \arg\min_{\tau} \Big\{ \underbrace{\| \underline{r}(t) - \underline{c}\,\underline{s}(t-\tau) \|^2}_{\text{error of reconstructed received signal } \underline{c}\,\underline{s}(t-\Delta t)} \Big\}$$

$$= \arg\min_{\tau} \Big\{ \int_{-\infty}^{+\infty} \big(\underline{r}(t) - \underline{c}\,\underline{s}(t-\tau)\big)^* \big(\underline{r}(t) - \underline{c}\,\underline{s}(t-\tau)\big)\, dt \Big\}$$

$$= \arg\min_{\tau} \Big\{ \int_{-\infty}^{+\infty} \Big(|\underline{r}(t)|^2 - 2\,\mathrm{Re}\big(\underline{c}^*\,\underline{s}^*(t-\tau)\,\underline{r}(t)\big) + |\underline{c}|^2\, |\underline{s}(t-\tau)|^2 \Big) dt \Big\}$$

| skip terms whose integral is independent of Δt

$$= \arg\max_{\tau} \Big\{ \int_{-\infty}^{+\infty} \mathrm{Re}\big(\underline{c}^*\,\underline{s}^*(t-\tau)\,\underline{r}(t)\big)\, dt \Big\}$$

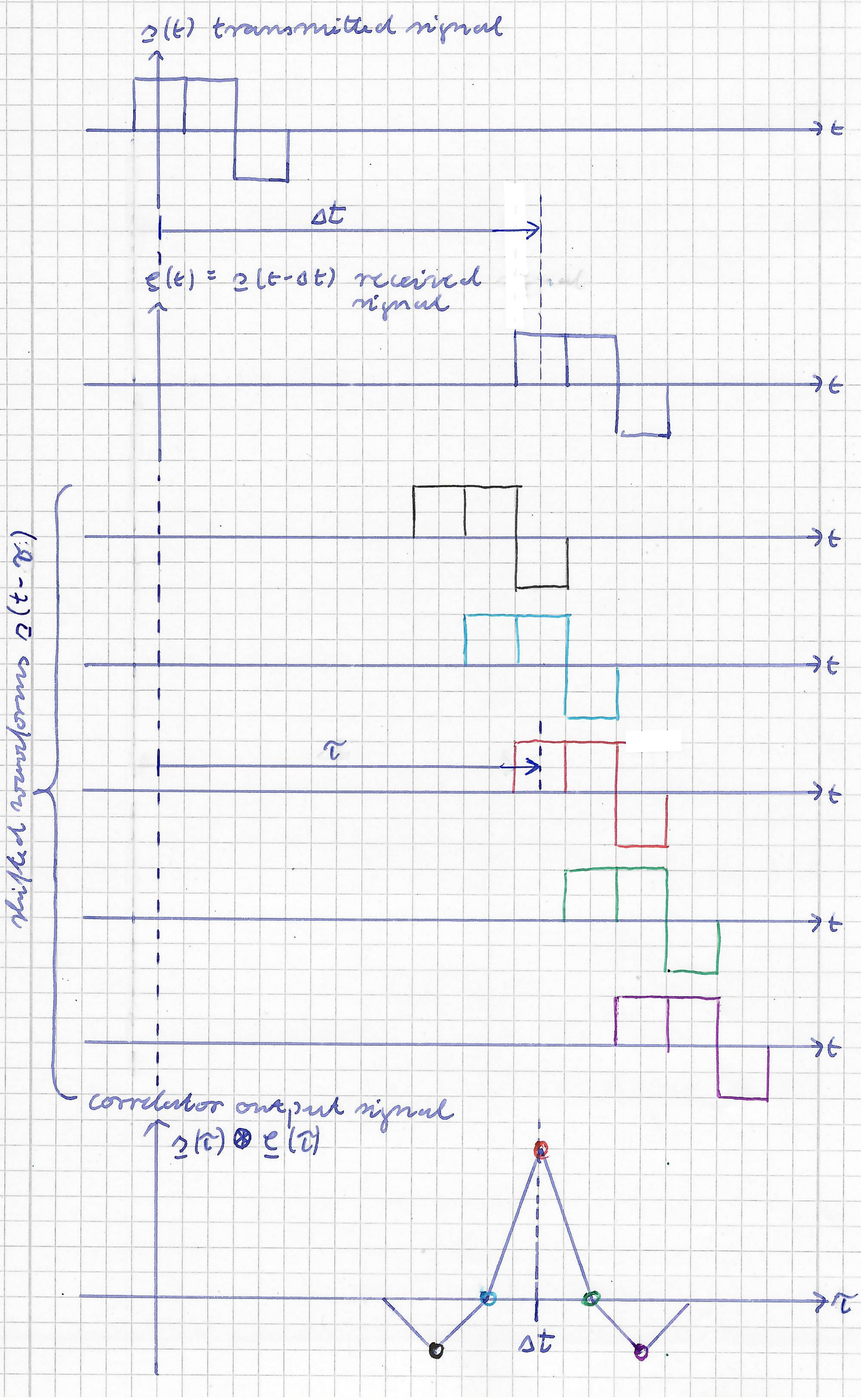
$$= \arg\max_{\tau} \Big\{ \mathrm{Re}\Big(\underbrace{\underline{c}^* \int_{-\infty}^{+\infty} \underline{s}^*(t-\tau)\,\underline{r}(t)\, dt}_{\text{real for the optimum } \underline{c}} \Big) \Big\}$$

$$= \arg\max_{\tau} \Big\{ \Big| \int_{-\infty}^{+\infty} \underline{s}^*(t-\tau)\,\underline{r}(t)\, dt \Big|^2 \Big\}$$

$$= \arg\max_{\tau} \Big\{ \underbrace{\Big| \underbrace{\int_{-\infty}^{+\infty} \underline{s}^*(t')\,\underline{r}(t'+\tau)\, dt'}_{\text{correlator } \underline{s}(\tau) \circledast \underline{r}(\tau)} \Big|^2}_{\text{incoherent correlator}} \Big\}$$

$\Rightarrow$ Correlate the received signal with the transmitted signal, take the magnitude squared, and search for the main peak.

Example: Correlator, Noise free Case



Point Spread Function

- noise free case

  $\underline{e}(t) \sim \underline{s}(t-\Delta t) = \underline{s}(t) * \delta(t-\Delta t)$

- correlator output signal

  $$\underline{s}(t) \circledast \underline{e}(t) \sim \underbrace{\underline{s}(t) \circledast \underline{s}(t)}_{\underline{\chi}(t)} * \delta(t-\Delta t)$$

- point spread function

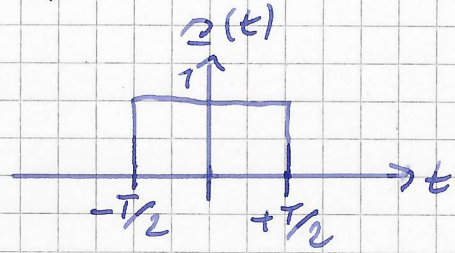
  $\underline{\chi}(t) = \underline{s}(t) \circledast \underline{s}(t) = \underline{R}^{E}_{ss}(t) = \langle \underline{s}(\tau), \underline{s}(\tau+t) \rangle$

  - equals the correlation function of the transmitted signal
  - point target $\delta(t-\Delta t)$ is spreaded by the point spread function $\underline{\chi}(t)$
  - for accurate delay estimates $\widehat{\Delta t}$ point spread functions $\underline{\chi}(t)$ with a narrow peak are required, the ideal point spread function would be a dirac pulse $\delta(t)$

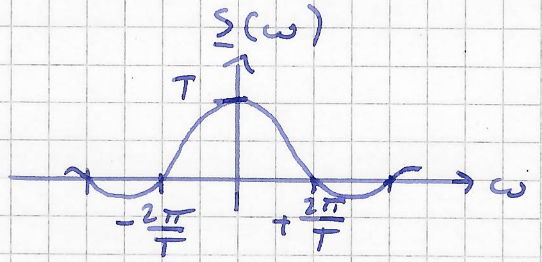
Example: Rectangular Pulse

- rectangular pulse of duration T

$$\underline{s}(t) = \mathrm{rect}\left(\frac{t}{T}\right)$$



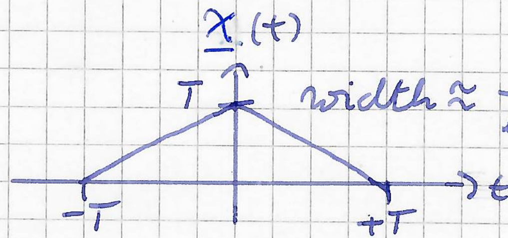
$$\underline{S}(\omega) = T\,\mathrm{si}\left(\frac{\omega T}{2}\right)$$



- point spread function

$$\underline{\chi}(t) = \underline{s}^*(-t) * \underline{s}(t) = T\Lambda\left(\frac{t}{T}\right) = \begin{cases} T\left(1-\frac{|t|}{T}\right) & , |t|<T \\ 0 & , \text{otherwise} \end{cases}$$

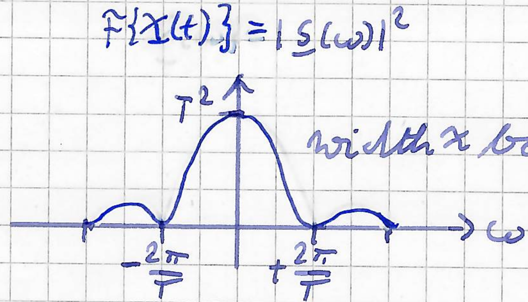
triangular pulse



width $\approx \frac{1}{\text{bandwidth}}$

$$\mathcal{F}\{\underline{\chi}(t)\} = \mathcal{F}\{\underline{s}^*(-t) * \underline{s}(t)\} = \underline{S}^*(\omega)\,\underline{S}(\omega) = |\underline{S}(\omega)|^2 = T^2\,\mathrm{si}^2\left(\frac{\omega T}{2}\right)$$

attention: the effective bandwidth of the rectangular pulse is infinite



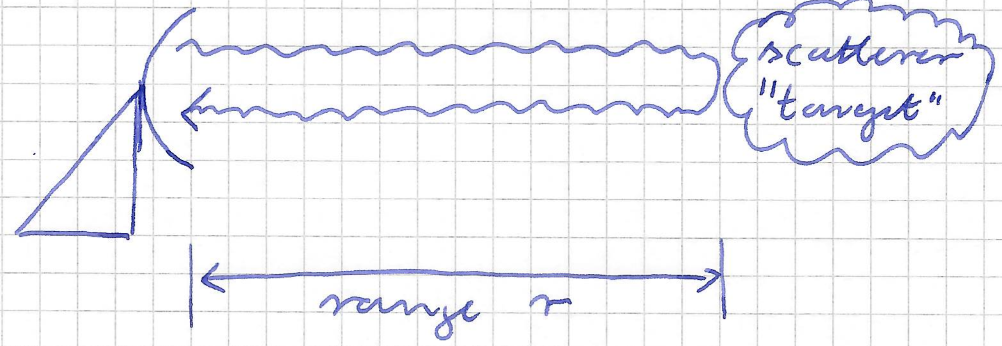
width $\approx$ bandwidth

- range resolution $\overset{\text{one way channel}}{=}$ c_0 time resolution

$$\approx \frac{2\pi\, c_0}{\text{bandwidth}}$$

(which in general may be much smaller than the duration)

Two Way Channel (Radar)



- delay $\Delta t = \frac{2r}{c_0}$

$\Rightarrow$ delay and range domain are the same except for a scaling by $2/c_0$

$\Rightarrow$ The only difference to range estimation in the one way channel is a factor of 2.