Deterministic Signals

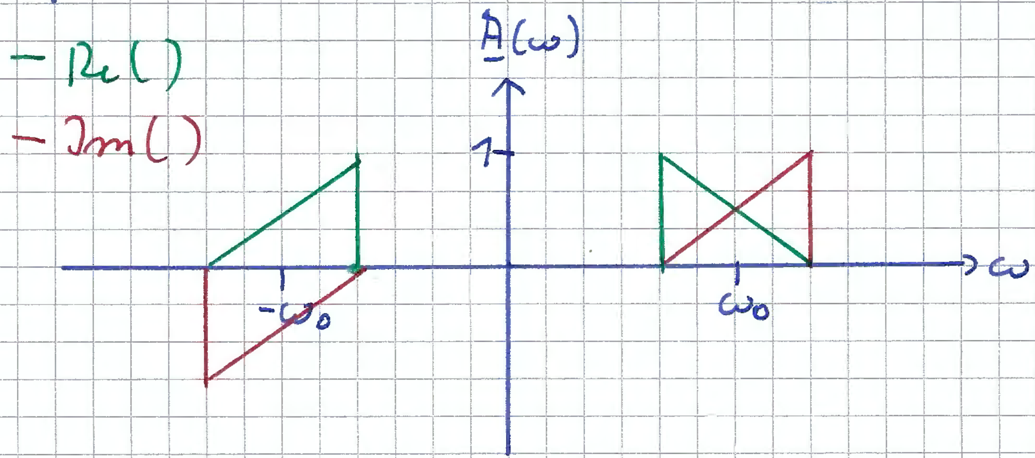
Bandpass Signals

- spectra of real signals $a(t)$ are symmetric:

$$\underline{A}(-\omega) = \underline{A}^*(\omega)$$

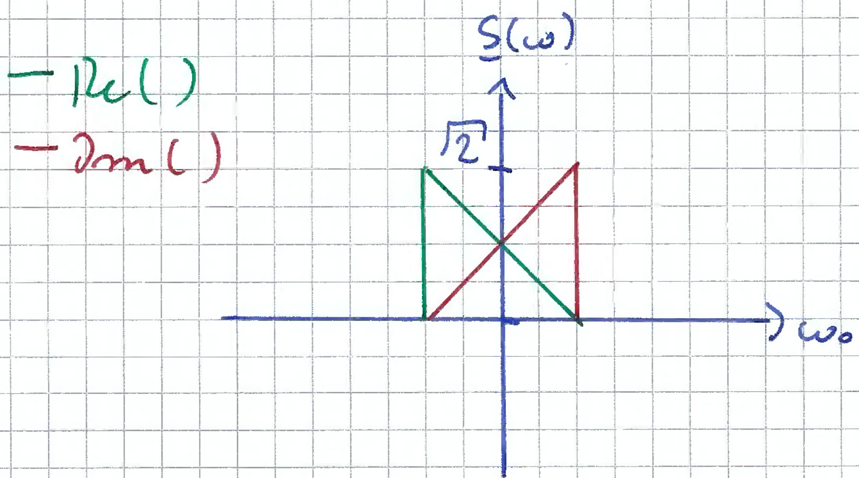
$\Rightarrow$ one half of the spectrum contains the complete information

- spectrum of a bandpass signal



- limit to positive frequencies
- shift by ω_0 to the left
- multiply by $\sqrt{2}$

- spectrum of the equivalent lowpass signal

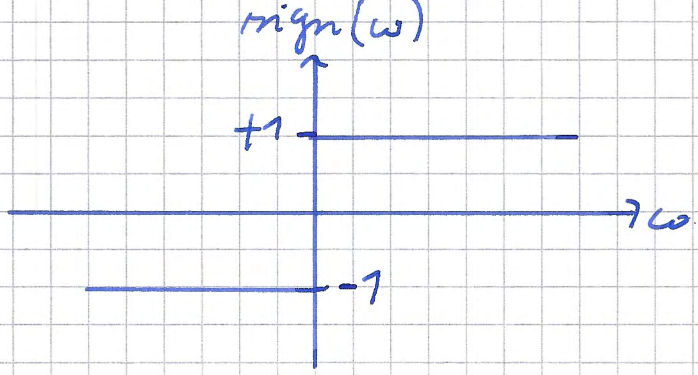


Notations

$a(t)$	bandpass signal
$z(t)$	equivalent lowpass signal, complex envelope
$\mathrm{Re}(z(t))$	in-phase component I-component
$\mathrm{Im}(z(t))$	quadrature component Q-component

Hilbert Transform

- signum function



- Hilbert transform

$$\mathcal{H}\{\underline{A}(\omega)\} = \hat{\underline{A}}(\omega) = -j\,\mathrm{sign}(\omega)\cdot\underline{A}(\omega)$$

$$\bullet\!\!-\!\!\circ \qquad\qquad \bullet\!\!-\!\!\circ \qquad\qquad \bullet\!\!-\!\!\circ$$

$$\mathcal{H}\{a(t)\} = \hat{a}(t) = \frac{1}{\pi t} * a(t)$$

Bandpass to Lowpass Transform

$$\underline{S}(\omega) = \frac{1}{\sqrt{2}}\left(1 + \operatorname{sign}(\omega+\omega_0)\right)\underline{A}(\omega+\omega_0)$$

$$= \frac{1}{\sqrt{2}}\underline{A}(\omega+\omega_0) + \frac{1}{\sqrt{2}}\underbrace{\operatorname{sign}(\omega+\omega_0)\,\underline{A}(\omega+\omega_0)}_{j\,\check{\underline{A}}(\omega+\omega_0)}$$

$$\underline{s}(t) = \frac{1}{\sqrt{2}}\,a(t)\,e^{-j\omega_0 t} + j\,\frac{1}{\sqrt{2}}\,\check{a}(t)\,e^{-j\omega_0 t}$$

$$= \frac{1}{\sqrt{2}}\left(\underbrace{a(t) + j\,\check{a}(t)}_{\text{analytic signal}}\right)e^{-j\omega_0 t}$$

Lowpass to Bandpass Transform

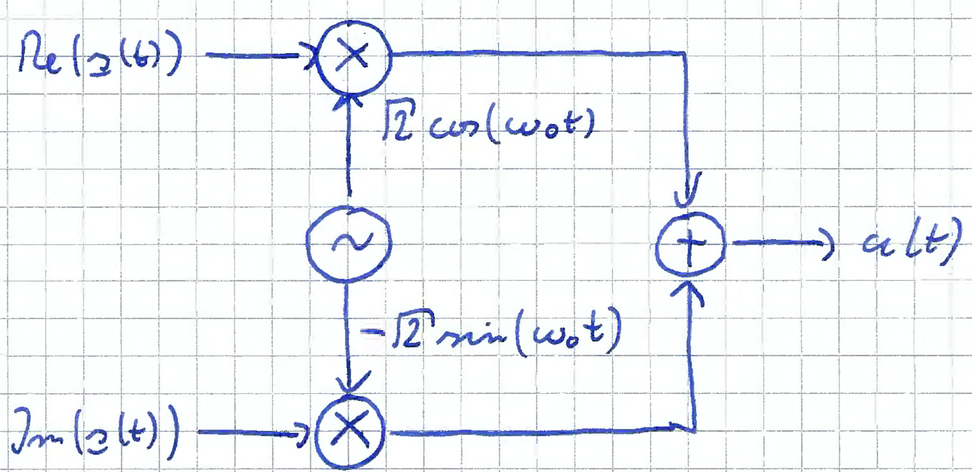
$$\underline{A}(\omega) = \frac{1}{\sqrt{2}} \underline{S}(\omega-\omega_0) + \frac{1}{\sqrt{2}} \underline{S}^*(-\omega-\omega_0)$$

$$a(t) = \frac{1}{\sqrt{2}} \underline{s}(t) e^{j\omega_0 t} + \frac{1}{\sqrt{2}} \underline{s}^*(t) e^{-j\omega_0 t}$$

$$= \sqrt{2}\,\mathrm{Re}\left(\underline{s}(t)\, e^{j\omega_0 t}\right)$$

$$= \sqrt{2}\,\mathrm{Re}(\underline{s}(t))\cos(\omega_0 t) - \sqrt{2}\,\mathrm{Im}(\underline{s}(t))\sin(\omega_0 t)$$

$\Rightarrow$ quadrature modulator



Phase Shift

phase shift by φ in the bandpass domain

$$\sqrt{2}\,\mathrm{Re}(\underline{z}(t))\cos(\omega_0 t+\varphi)$$
$$-\sqrt{2}\,\mathrm{Im}(\underline{z}(t))\sin(\omega_0 t+\varphi)$$
$$=\sqrt{2}\,\mathrm{Re}\left(\underline{z}(t)\,e^{j(\omega_0 t+\varphi)}\right)$$
$$=\sqrt{2}\,\mathrm{Re}\Big(\underbrace{\underline{z}(t)\,e^{j\varphi}}\,e^{j\omega_0 t}\Big)$$

lowpass equivalent of the phase shifted bandpass signal, rotation in the complex plane

Frequency Shift

- frequency shift by $\Delta\omega$ corresponds to a linearly increasing phase shift

$$\varphi = \Delta\omega t$$

- time domain

$$\underline{s}(t)\, e^{j\Delta\omega t}$$

- frequency domain

$$\mathcal{F}\{\underline{s}(t)\, e^{j\Delta\omega t}\} = \underline{S}(\omega - \Delta\omega)$$

Time Shift

- time shift by Δt in the bandpass domain

$$a(t-\Delta t) = \sqrt{2}\,\mathrm{Re}\left(\underline{s}(t-\Delta t)\, e^{j\omega_0 (t-\Delta t)}\right)$$

$$= \sqrt{2}\,\mathrm{Re}\left(\underbrace{\underline{s}(t-\Delta t)\, e^{-j\omega_0 \Delta t}}_{\substack{\text{lowpass equivalent}\\ \text{of the time shifted}\\ \text{bandpass signal}}}\, e^{j\omega_0 t}\right)$$

- frequency domain

$$\mathcal{F}\{\underline{s}(t-\Delta t)\, e^{-j\omega_0 \Delta t}\} = \underline{S}(\omega)\, e^{-j\omega \Delta t}\, e^{-j\omega_0 \Delta t}$$

$$= \underline{S}(\omega)\, e^{-j(\omega+\omega_0)\Delta t}$$

$\Rightarrow$ dual to frequency shift (additional factor $e^{-j\omega_0 \Delta t}$)

Narrowband Approximation

relatively small time shift

$$\Delta t \ll 1/\Omega$$

$$\Rightarrow a(t-\Delta t) \approx \sqrt{2}\, \mathrm{Re}\left(\underline{a}(t)\, e^{-j\omega_0 \Delta t}\, e^{j\omega_0 t} \right)$$

A small time shift by Δt in the bandpass domain corresponds to a phase shift

$$\varphi = -\omega_0 \Delta t$$

in the bandpass domain, which corresponds to a rotation in the complex plane in the lowpass domain!

Energy

$$E = \int_{-\infty}^{+\infty} a^2(t)\,dt$$ by definition

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{|\underline{A}(\omega)|^2}_{\text{energy density spectrum}} d\omega$$ using Parseval's theorem

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underbrace{|\underline{S}(\omega)|^2}_{\text{energy density spectrum}} d\omega$$ see bandpass and equivalent lowpass spectra

$$= \int_{-\infty}^{+\infty} |\underline{s}(t)|^2\,dt$$ using Parseval's theorem

Effective Duration of a Signal

- mean occurrence time

$$\bar{t} = \frac{\int_{-\infty}^{+\infty} t \, |\underline{s}(t)|^2 \, dt}{\int_{-\infty}^{+\infty} |\underline{s}(t)|^2 \, dt}$$

- effective duration

$$T_e = \sqrt{\frac{\int_{-\infty}^{+\infty} (t-\bar{t})^2 \, |\underline{s}(t)|^2 \, dt}{\int_{-\infty}^{+\infty} |\underline{s}(t)|^2 \, dt}}$$

Example: Effective Duration of a Gaussian Pulse

- Gaussian pulse

$$s(t) = \sqrt[4]{\frac{1}{2\pi T^2}} \, e^{-\frac{t^2}{4T^2}}$$

- energy

$$E = \int_{-\infty}^{+\infty} |s(t)|^2 \, dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,T} e^{-\frac{t^2}{2T}} \, dt \qquad \Big| \int_{-\infty}^{+\infty} e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{a}$$

$= 1$ energy normalized!

- mean occurrence time

$\bar{t} = 0$ due to symmetry

- effective duration

$$T_e = \sqrt{\int_{-\infty}^{+\infty} t^2 \, |s(t)|^2 \, dt}$$

$$= \sqrt{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,T} t^2 e^{-\frac{t^2}{2T^2}} \, dt} \qquad \Big| \int_{-\infty}^{+\infty} x^2 e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{2a^3}$$

$= T$

Effective Bandwidth of a Signal

- center frequency

$$\bar{\omega} = \frac{\int_{-\infty}^{+\infty} \omega \, |\underline{S}(\omega)|^2 \, d\omega}{\int_{-\infty}^{+\infty} |\underline{S}(\omega)|^2 \, d\omega}$$

- effective bandwith

$$\Omega_C = \sqrt{\frac{\int_{-\infty}^{+\infty} (\omega - \bar{\omega})^2 \, |\underline{S}(\omega)|^2 \, d\omega}{\int_{-\infty}^{+\infty} |\underline{S}(\omega)|^2 \, d\omega}}$$

Example: Effective Bandwidth of a Gaussian Pulse

- Fourier transform of a Gaussian pulse is a Gaussian pulse

$$\mathcal{F}\{\underline{s}(t)\} = \mathcal{F}\left\{\sqrt[4]{\frac{1}{2\pi T^2}}\, e^{-\frac{t^2}{4T^2}}\right\}$$

$$= \sqrt[4]{8\pi T^2}\, e^{-T^2\omega^2} = \underline{S}(\omega)$$

- energy

$$E = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\underline{S}(\omega)|^2\, d\omega = \int_{-\infty}^{+\infty} |\underline{s}(t)|^2\, dt = 1$$

- center frequency

$\bar{\omega} = 0$ due to symmetry

- effective bandwidth

$$\Omega_e = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega^2\, |\underline{S}(\omega)|^2\, d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \sqrt{8\pi T^2}\, \omega^2\, e^{-2T^2\omega^2}\, d\omega}$$

$$\left|\ \int_{-\infty}^{+\infty} x^2 e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{2a^3}\right.$$

$$= \frac{1}{2\pi}$$

Time-Bandwidth Product

- for simplicity

$$\bar{t} = 0$$

$$\bar{\omega} = 0$$

$$E = \int_{-\infty}^{+\infty} |\underline{s}(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\underline{S}(\omega)|^2 \, d\omega = 1$$

- effective duration

$$T_e^2 = \int_{-\infty}^{+\infty} t^2 \, |\underline{s}(t)|^2 \, dt$$

- effective bandwidth

$$\Omega_e^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega^2 \, |\underline{S}(\omega)|^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |j\omega \, \underline{S}(\omega)|^2 \, d\omega$$

using Parseval's theorem

$$= \int_{-\infty}^{+\infty} \left| \frac{\partial \underline{s}(t)}{\partial t} \right|^2 dt$$

• time-bandwidth product

$$T_e^2 \Omega_e^2 = \int_{-\infty}^{+\infty} t^2 |\underline{s}(t)|^2 dt \int_{-\infty}^{+\infty} \left| \frac{\partial \underline{s}(t)}{\partial t} \right|^2 dt$$

$$= \int_{-\infty}^{+\infty} | t \underline{s}(t)|^2 dt \int_{-\infty}^{+\infty} \left| \frac{\partial \underline{s}(t)}{\partial t} \right|^2 dt$$

using Schwarz inequality

$$\geq \left| \int_{-\infty}^{+\infty} t \underline{s}(t) \frac{\partial \underline{s}^*(t)}{\partial t} dt \right|^2$$

$$\geq \left(\int_{-\infty}^{+\infty} \mathrm{Re}\left(t \underline{s}(t) \frac{\partial \underline{s}^*(t)}{\partial t} \right) dt \right)^2$$

$$= \frac{1}{4} \left(\int_{-\infty}^{+\infty} t \underbrace{\left(\underline{s}(t) \frac{\partial \underline{s}^*(t)}{\partial t} + \underline{s}^*(t) \frac{\partial \underline{s}(t)}{\partial t} \right)}_{\frac{\partial}{\partial t}\left\{ \underline{s}(t) \underline{s}^*(t) \right\} = \frac{\partial}{\partial t}\left\{ |\underline{s}(t)|^2 \right\}} dt \right)^2$$

integrating by parts

$$= \frac{1}{4} \left(\underbrace{\left[t |\underline{s}(t)|^2 \right]_{-\infty}^{+\infty}}_{0} - \underbrace{\int_{-\infty}^{+\infty} |\underline{s}(t)|^2 dt}_{E = 1} \right)^2$$

$$= \frac{1}{4}$$

$$\Rightarrow T_e \Omega_e \geq \frac{1}{2} \approx 1$$

Example: Time-Bandwidth Product of a Gaussian Pulse

$$T_e \Omega_e = \frac{1}{2}$$

The Gaussian pulse has the smallest possible time-bandwidth product!

Heisenberg Uncertainty Principle

Using the momentum

$$p = \frac{h\,\omega}{2\pi\, c_0}$$

of a photon one obtains

$$\Delta z \,\Delta p = T_E \cdot c_0 \, \frac{h\,\Omega_E}{2\pi\, c_0}$$

$$= \frac{h}{2\pi}\, T_E\, \Omega_E$$

$$\geq 0{,}5\, \frac{h}{2\pi}$$

Planck constant

$$h = 6{,}626\,070\,15 \cdot 10^{-34}\ \mathrm{Js}$$