Systems

Filtering of Signals

- linear time invariant system with impulse response $\underline{h}(t)$

$$\longrightarrow \boxed{\underline{h}(t)} \longrightarrow$$

$$\underline{s}(t) \quad \circ\!\!-\!\!\bullet \quad \underline{S}(\omega)$$

$$\underline{e}(t) = \underline{h}(t) * \underline{s}(t) \quad \circ\!\!-\!\!\bullet \quad \underline{E}(\omega) = \underline{H}(\omega)\,\underline{S}(\omega)$$

- transfer function

$$\underline{H}(\omega) = \mathcal{F}\{\underline{h}(t)\}$$

Filtering of Noise, Time Domain

$$\underline{n}(t) \longrightarrow \boxed{\underline{h}(t)} \longrightarrow \underline{m}(t)$$

$$\underline{R}_{mm}(t) = E\{\underline{m}^*(\tau)\,\underline{m}(\tau+t)\}$$

$$= E\left\{\int_{-\infty}^{+\infty} \underline{h}^*(\tau')\,\underline{n}^*(\tau-\tau')\,d\tau' \int_{-\infty}^{+\infty} \underline{h}(\tau'')\,\underline{n}(\tau+t-\tau'')\,d\tau''\right\}$$

$$= E\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underline{h}^*(\tau')\,\underline{h}(\tau'')\,\underline{n}^*(\tau-\tau')\,\underline{n}(\tau+t-\tau'')\,d\tau'\,d\tau''\right\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underline{h}^*(\tau')\,\underline{h}(\tau'+\tau''')\,\underbrace{E\{\underline{n}^*(\tau-\tau')\,\underline{n}(\tau+t-\tau'-\tau''')\}}_{\underline{R}_{nn}(t-\tau''')}\,d\tau'\,d\tau'''$$

$$= \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underline{h}^*(\tau')\,\underline{h}(\tau'+\tau''')\,d\tau'}_{\text{autocorrelation function}} \quad \underline{R}_{nn}(t-\tau''')\,d\tau'''$$

$$\underline{R}^E_{hh}(\tau''') = \underline{h}(\tau''') \circledast \underline{h}(\tau''')$$
$$= \underline{h}^*(-\tau''') * \underline{h}(\tau''')$$

$$= \underline{R}^E_{hh}(t) * \underline{R}_{nn}(t)$$

Filtering of Noise, Frequency Domain

- energy density spectrum

$$\hat{\mathcal{F}}\{ \underline{R}^{E}_{hh}(t) \} = \hat{\mathcal{F}}\{ \underline{h}^{*}(-t) * \underline{h}(t) \}$$

$$= \underline{H}^{*}(\omega) \; \underline{H}(\omega)$$

$$= | \underline{H}(\omega) |^{2}$$

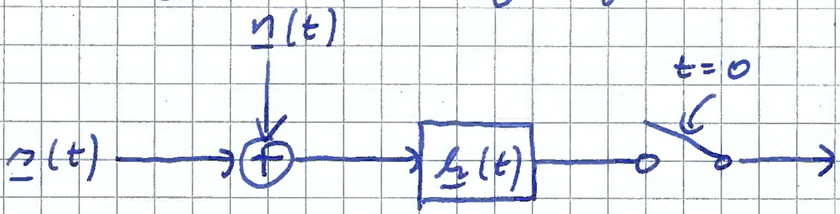
- power density spectrum of the noise

$$S_{mm}(\omega) = \hat{\mathcal{F}}\{ \underline{R}^{E}_{hh}(t) * \underline{R}_{nn}(t) \}$$

$$= | \underline{H}(\omega) |^{2} \; S_{nn}(\omega)$$

Signal-to-Noise Ratio (SNR)

- filtering of a noisy signal



- signal power at filter output

$$S = \left| \underline{h}(t) * \underline{s}(t) \right|^2_{t=0}$$

$$= \left| \mathcal{F}^{-1}\left\{ \underline{H}(\omega)\, \underline{S}(\omega) \right\} \right|^2_{t=0}$$

$$= \left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{H}(\omega)\, \underline{S}(\omega)\, d\omega \right|^2$$

- noise power at filter output

$$N = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{mm}(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \underline{H}(\omega) \right|^2 S_{nn}(\omega)\, d\omega$$

- signal-to-noise ratio

$$\gamma = \frac{S}{N} = \frac{\left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{H}(\omega)\, \underline{S}(\omega)\, d\omega \right|^2}{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \underline{H}(\omega) \right|^2 S_{nn}(\omega)\, d\omega}$$

- special case: white noise $S_{nn}(\omega) = N_0$

$$\gamma = \frac{\left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{H}(\omega)\, \underline{S}(\omega)\, d\omega \right|^2}{\frac{N_0}{2\pi} \int_{-\infty}^{+\infty} \left| \underline{H}(\omega) \right|^2 d\omega}$$

Matched Filter

Find the transfer function $\underline{H}(\omega)$ which maximizes the SNR γ!

- Schwarz inequality

$$\gamma = \frac{\left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{H}(\omega)\, \underline{S}(\omega)\, d\omega \right|^2}{\frac{N_0}{2\pi} \int_{-\infty}^{+\infty} |\underline{H}(\omega)|^2\, d\omega}$$

$$\leq \frac{\cancel{\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\underline{H}(\omega)|^2 d\omega}\ \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\underline{S}(\omega)|^2 d\omega}{N_0\, \cancel{\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\underline{H}(\omega)|^2 d\omega}}$$

- equality, i.e., maximum SNR for

$$\underline{H}(\omega) \sim \underline{S}^*(\omega)$$

$$\underline{h}(t) \sim \underline{s}^*(-t)$$

$\Rightarrow$ matched filter performs a correlation with $\underline{s}(t)$

- maximum SNR

$$\gamma_{max} = \frac{\frac{1}{2\pi} \int_{-\infty}^{+\infty} |\underline{S}(\omega)|^2 d\omega}{N_0} = \frac{\text{energy of the signal}}{\text{noise power density}}$$

$\Rightarrow$ the energy, not the power of the signal $\underline{s}(t)$ is crucial