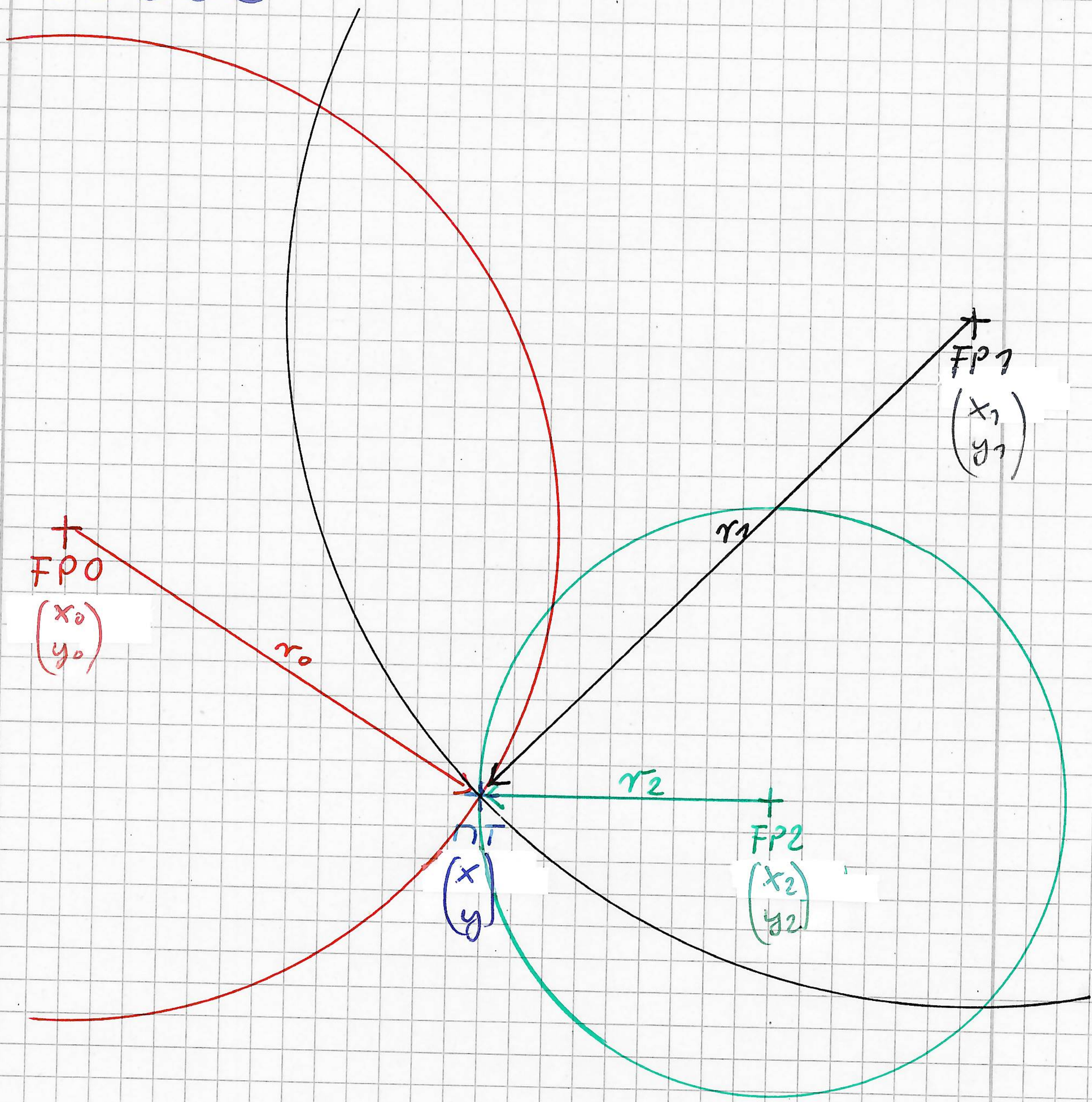
Time of Arrival (TOA)

Scenario



FPs are the centers of the spheres

Assumptions

- for simplicity 2D-scenario, extension to 3D-scenarios straightforward
- K fixed points (FPs) at known positions $\begin{pmatrix} x_k \\ y_k \end{pmatrix}$, $k = 0 \dots K-1$, e.g., satellites in GPS
- without loss of generality: choose coordinates such that FP0 is at the origin, i.e., $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
- measure ranges r_k, $k = 0 \dots K-1$ (equivalent to delays, "Time of Arrival", TOA)

  $\Rightarrow$ requires synchronized clocks at Tx and Rx
- task: determine the position $\begin{pmatrix} x \\ y \end{pmatrix}$ of the mobile terminal (MT)
- here: point positioning (single set of measurements)
- later: tracking (sequence of measurements $\rightarrow$ trajectory)

Time of Arrival (TOA)

- each measured range

$$r_n = \sqrt{(x-x_n)^2 + (y-y_n)^2}$$

defines a sphere of possible MT positions

=> find the intersection point of the spheres, various algorithms to solve this nonlinear (in general overdetermined) system of equations

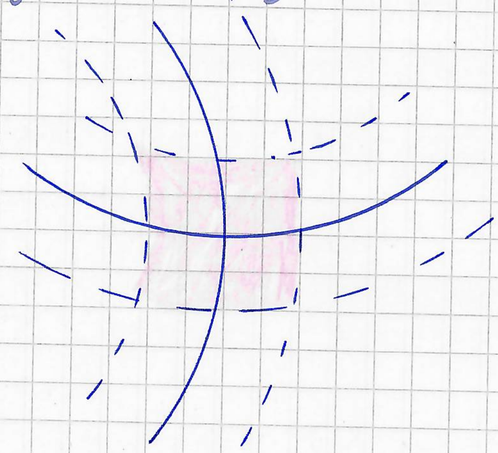
- in 2D scenarios 2 unknowns x and y

- 2 FPs => 2 equations but in general two intersection points, ambiguity might be resolved by some side information

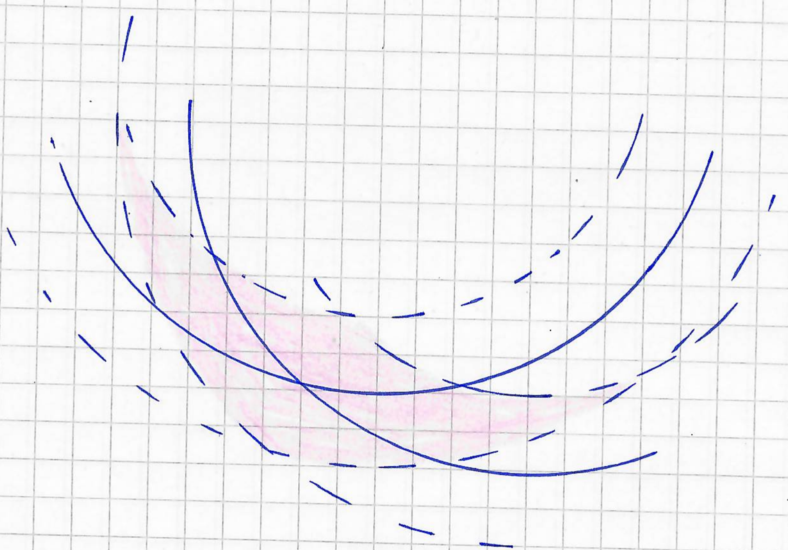
- ≥ 3 FPs => one unique intersection point (if no measurement errors)

Measurement Errors

- typically no common intersection point of all spheres
  => pseudo solution required
- localization error (error propagation) depends on FP constellation
  => design of good constellations



good constellation



bad constellation

Analytical Method

requires $K=2$ FPs in 2D scenarios

(0) $$r_0^2 = x^2 + y^2 \qquad (x_0 = 0,\ y_0 = 0)$$

(1) $$r_1^2 = (x-x_1)^2 + (y-y_1)^2$$
$$= x^2 - 2x x_1 + x_1^2 + y^2 - 2y y_1 + y_1^2$$

(1) − (0) $$r_1^2 - r_0^2 = -2x x_1 + x_1^2 - 2y\, y_1 + y_1^2$$

(3) $$x = \underbrace{-\frac{y_1}{x_1}}_{A}\, y + \underbrace{\frac{r_0^2 - r_1^2 - x_1^2 - y_1^2}{2x_1}}_{B}$$

substitute this into (0) ⇒ quadratic equation for y

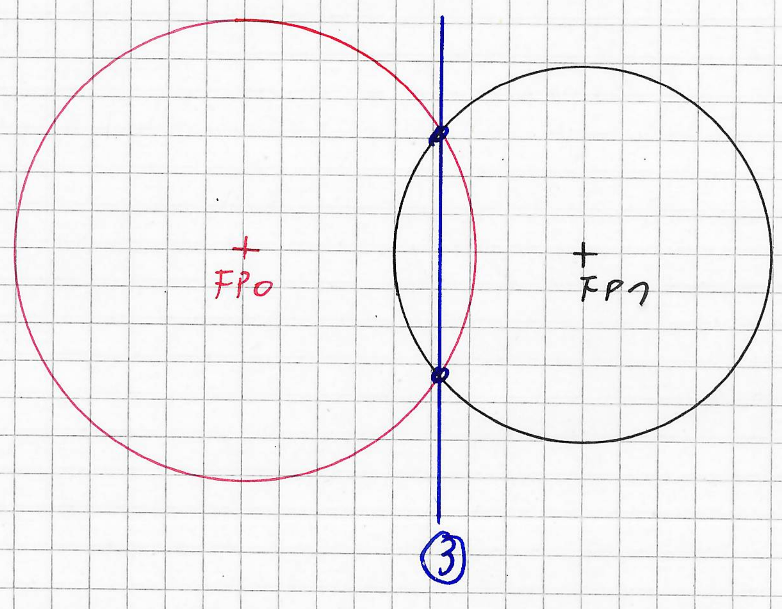
$$(A^2+1)\, y^2 + 2AB\, y + B^2 - r_0^2 = 0$$

$$\Rightarrow y = -\frac{AB}{A^2+1} \pm \sqrt{\left(\frac{AB}{A^2+1}\right)^2 - \frac{B^2 - r_0^2}{A^2+1}}$$

in general two solutions (as expected)

using (3) $\quad x = Ay + B$

remark: (3) defines a straight line of possible MT positions



Least Squares Method

- requires $K \geq 3$ FPs in 2D scenarios

- $K-1$ linear equations for the two variables x and y can be obtained:

$$(0) \qquad r_0^2 = x^2 + y^2 \qquad (x_0 = 0,\ y_0 = 0)$$

$$\vdots$$

$$(k) \qquad r_k^2 = (x - x_k)^2 + (y - y_k)^2$$
$$= x^2 - 2 x x_k + x_k^2 + y^2 - 2 y y_k + y_k^2$$

$$\vdots$$

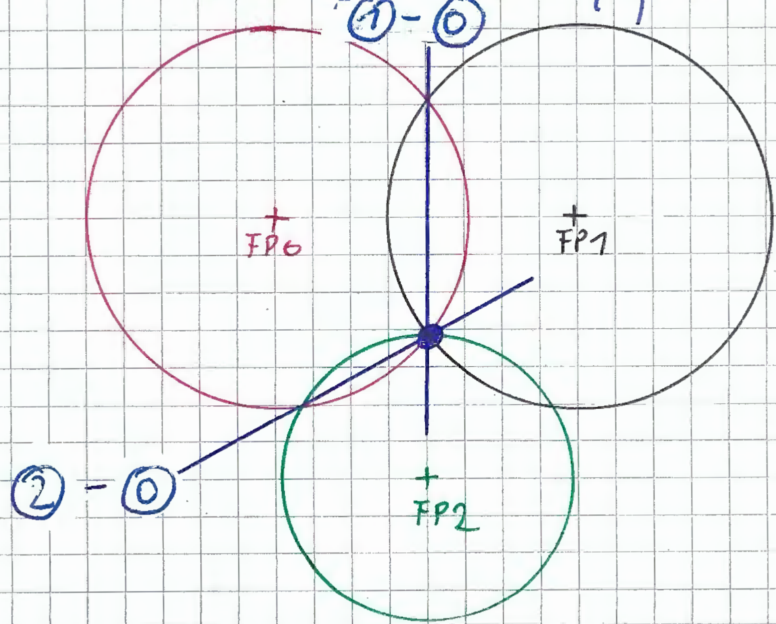
$$\Rightarrow (k) - (0) \qquad r_k^2 - r_0^2 = -2 x x_k - 2 y y_k + x_k^2 + y_k^2$$

- set up an (over-) determined linear system of equations

$$\underbrace{2 \begin{pmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_{K-1} & y_{K-1} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} x_1^2 + y_1^2 - r_1^2 + r_0^2 \\ \vdots \\ x_{K-1}^2 + y_{K-1}^2 - r_{K-1}^2 + r_0^2 \end{pmatrix}}_{b}$$

$$A \cdot x = b$$

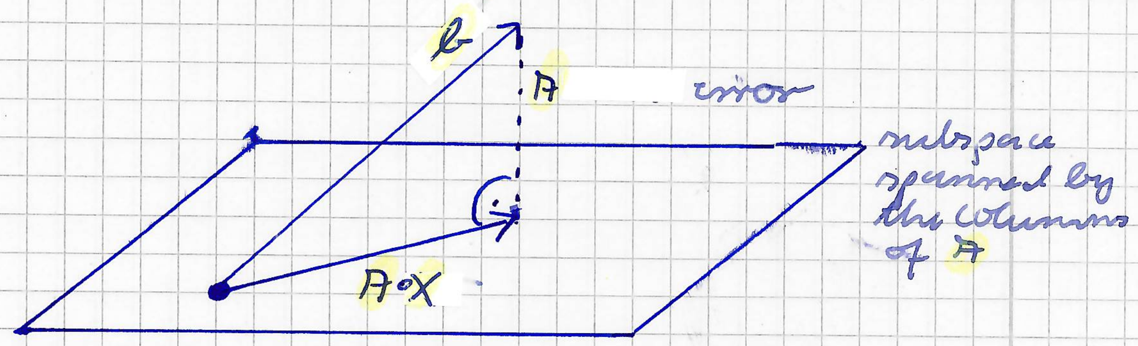
- remark: each of the linear equations $(k) - (0)$ defines a straight line of possible MT positions



find the intersection point of the straight lines!

Least Squares Pseudosolution of Overdetermined Linear Systems of Equations

$A \cdot x = b$



minimize the error $\|A \cdot x - b\|^2$

the error becomes minimum if the error vector $(A \cdot x - b)$ is orthogonal to the subspace spanned by the columns of A

$$\Rightarrow \quad A^T \cdot (A \cdot x - b) = 0$$

$$A^T \cdot A \cdot x - A^T \cdot b = 0$$

$$A^T \cdot A \cdot x = A^T \cdot b$$

$$x = (A^T \cdot A)^{-1} \cdot A^T \cdot b$$

Gauss-Newton Method

- idea: linearize the range equations around an initial estimate $\begin{pmatrix} x(n) \\ y(n) \end{pmatrix}$

$$r_k = \sqrt{(x-x_k)^2 + (y-y_k)^2}$$

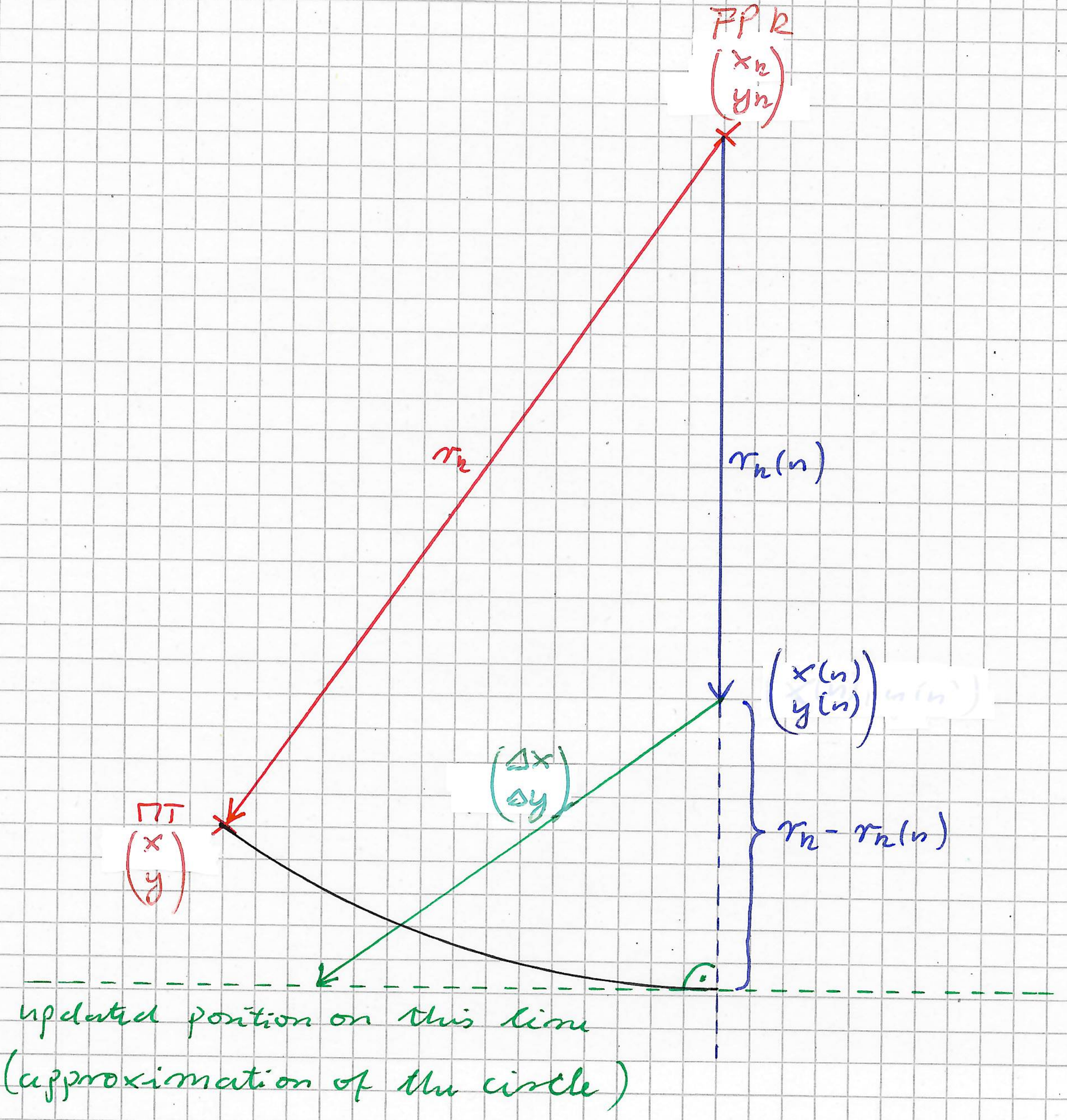
$$\approx \underbrace{\sqrt{(x(n)-x_k)^2 + (y(n)-y_k)^2}}_{r_k(n),\ \text{range computed from } \begin{pmatrix} x(n) \\ y(n) \end{pmatrix}}$$

$$+ \frac{\partial r_k}{\partial x} \underbrace{(x - x(n))}_{\Delta x(n)}$$

$$+ \frac{\partial r_k}{\partial y} \underbrace{(y - y(n))}_{\Delta y(n)}$$

$$= r_k(n) + \frac{x(n)-x_k}{r_k(n)} \Delta x(n) + \frac{y(n)-y_k}{r_k(n)} \Delta y(n)$$

$$= r_k(n) + \underbrace{\left\langle \underbrace{\begin{pmatrix} \frac{x(n)-x_k}{r_k(n)} \\ \frac{y(n)-y_k}{r_k(n)} \end{pmatrix}}_{\substack{\text{unit vector pointing} \\ \text{from FPk towards} \\ \text{initial estimate of} \\ \text{MT position}}}, \underbrace{\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}}_{\Delta(n)} \right\rangle}_{\text{projection}}$$



FPK
$\begin{pmatrix} x_k \\ y_k \end{pmatrix}$
r_k
$r_k(n)$
$\begin{pmatrix} x(n) \\ y(n) \end{pmatrix}$
$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$
MT
$\begin{pmatrix} x \\ y \end{pmatrix}$
$r_k - r_k(n)$
updated position on this line
(approximation of the circle)

- set up an (over-) determined linear system of equations for computing the correction $\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$, requires $K \geq 2$ FPs in 2D scenarios

$$\underbrace{\begin{pmatrix} \frac{x(n)-x_0}{r_0(n)} & \frac{y(n)-y_0}{r_0(n)} \\ \vdots & \vdots \\ \frac{x(n)-x_{K-1}}{r_{K-1}(n)} & \frac{y(n)-y_{K-1}}{r_{K-1}(n)} \end{pmatrix}}_{A(n)} \cdot \underbrace{\begin{pmatrix} \Delta x(n) \\ \Delta y(n) \end{pmatrix}}_{\Delta(n)} = \underbrace{\begin{pmatrix} r_0 - r_0(n) \\ \vdots \\ r_{K-1} - r_{K-1}(n) \end{pmatrix}}_{b(n)}$$

- least squares pseudosolution

$$\Delta(n) = \left(A^T(n) \cdot A(n) \right)^{-1} \cdot A^T(n) \cdot b(n)$$

- obtain updated estimates

$$x(n+1) = x(n) + \Delta x(n)$$
$$y(n+1) = y(n) + \Delta y(n)$$

- repeat this procedure iteratively until desired accuracy / stop criterion is reached, typically good convergence

Dilution of Precision (DOP)

- definition

$$DOP = \frac{\text{standard deviation of position error}}{\text{standard deviation of range error}}$$

- assumption: measurement errors are so small that the linearized model (see Gauss-Newton method) holds in very good approximation

- let $\begin{pmatrix} x \\ y \end{pmatrix}$ be the true position of the MT and r_k be the true range

  $\Rightarrow$ then the measured range which seems to be due to a position $\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$ of the MT reads

$$\tilde{r}_k \approx r_k + \frac{x - x_k}{r_k} \underbrace{(\tilde{x} - x)}_{\Delta x} + \frac{y - y_k}{r_k} \underbrace{(\tilde{y} - y)}_{\Delta y}$$

- set up an (over-) determined linear system of equations

$$\underbrace{\begin{pmatrix} \frac{x - x_0}{r_0} & \frac{y - y_0}{r_0} \\ \vdots & \vdots \\ \frac{x - x_{K-1}}{r_{K-1}} & \frac{y - y_{K-1}}{r_{K-1}} \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}}_{\Delta} = \underbrace{\begin{pmatrix} \tilde{r}_0 - r_0 \\ \vdots \\ \tilde{r}_{K-1} - r_{K-1} \end{pmatrix}}_{b}$$

Δ: position error

b: measurement errors, independent and identically distributed, zero mean, standard deviation σ

- least squares pseudosolution

$$\Delta = (A^T \cdot A)^{-1} \cdot A \cdot b$$

- correlation matrix of the position error

$$E\{\Delta \cdot \Delta^T\} = E\{(A^T \cdot A)^{-1} \cdot A^T \cdot b \cdot b^T \cdot A \cdot (A^T \cdot A)^{-1}\}$$

$$= (A^T \cdot A)^{-1} \cdot A^T \cdot \underbrace{E\{b \cdot b^T\}}_{\sigma^2 E} \cdot A \cdot (A^T \cdot A)^{-1}$$

$$= \sigma^2 (A^T \cdot A)^{-1} \cdot A^T \cdot A \cdot (A^T \cdot A)^{-1}$$

$$= \sigma^2 (A^T \cdot A)^{-1}$$

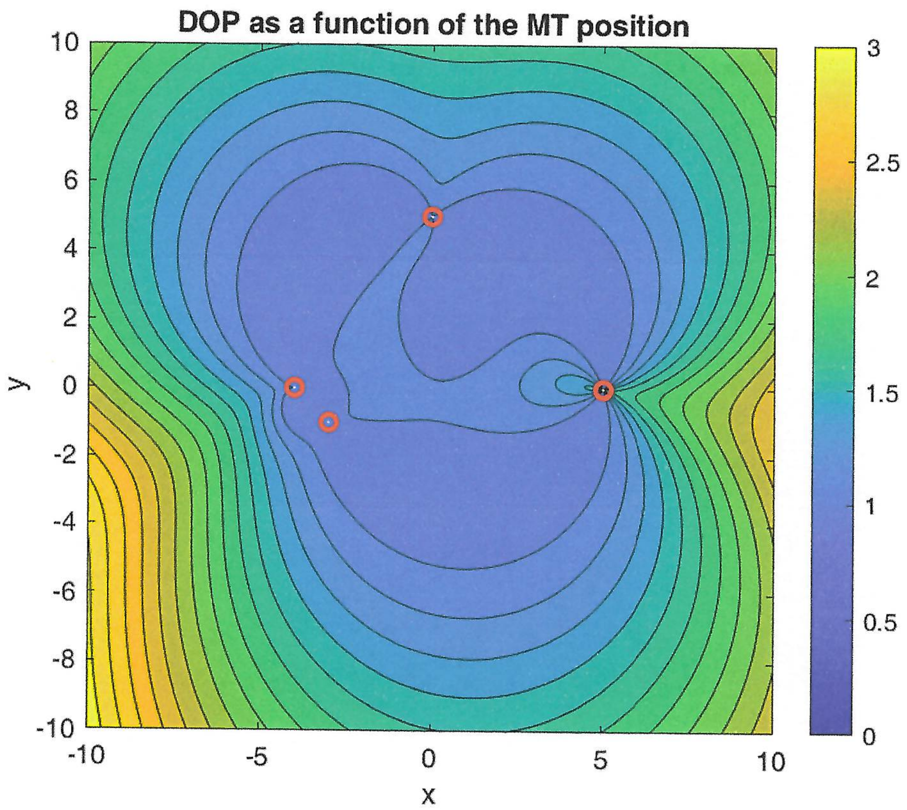
- standard deviation of the position error

$$\sigma_p = \sqrt{\operatorname{trace}(\sigma^2 (A^T \cdot A)^{-1})}$$

$$= \sigma \sqrt{\operatorname{trace}(A^T \cdot A)^{-1}}$$

$$\Rightarrow DOP = \sqrt{\operatorname{trace}(A^T \cdot A)^{-1}}$$

compare: Cramer Rao lower bound (CRLB) for the case of independent and identically zero mean Gaussian distributed range measurement errors with variance σ^2



DOP as a function of the MT position
10
8
6
4
2
0
-2
-4
-6
-8
-10
y
-10
-5
0
5
10
x
3
2.5
2
1.5
1
0.5
0

○ K = 4 FPs