Ambiguity Function

Definition of the Ambiguity Function

goal: performance assessment of radar concepts based on their characteristic waveforms $\underline{s}(t)$

- point spread function, complex ambiguity function
  cross-correlation function of the waveform $\underline{s}(t)$ and the frequency shifted waveform $\underline{s}(t)e^{-j\nu t}$

$$\underline{X}(t,\nu) = \underline{s}(t) \circledast \underline{s}(t)e^{-j\nu t}$$

$$= \langle \underline{s}(\tau), \underline{s}(\tau+t)e^{-j\nu(\tau+t)} \rangle$$

$$= \int_{-\infty}^{+\infty} \underline{s}^*(\tau)\, \underline{s}(\tau+t)\, e^{-j\nu(\tau+t)}\, d\tau$$

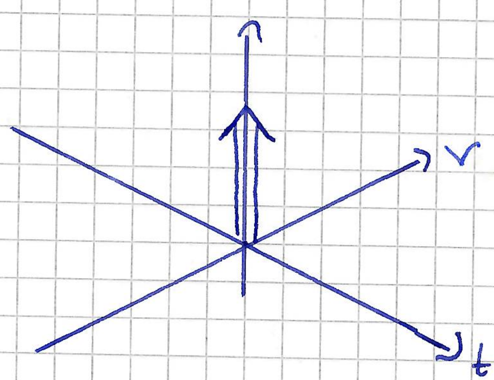
$$= \int_{-\infty}^{+\infty} \underline{s}^*(\tau'-t)\, \underline{s}(\tau')\, e^{-j\nu\tau'}\, d\tau'$$

- ambiguity function

$$|\underline{X}(t,\nu)| = \left| \int_{-\infty}^{+\infty} \underline{s}^*(\tau-t)\, \underline{s}(\tau)\, e^{-j\nu\tau}\, d\tau \right|$$

- "ideal" ambiguity function

$$|\underline{X}(t,\nu)| = \delta(t)\,\delta(\nu)$$



Unfortunately, no such waveforms exist.

Frequency Domain

- Fourier transform

$$\mathcal{F}\{\underline{\chi}(t,\nu)\} = \mathcal{F}\{\underline{s}(t) \circledast \underline{s}(t)e^{-j\nu t}\}$$

$$= \mathcal{F}\{\underline{s}^*(-t) * \underline{s}(t)e^{-j\nu t}\}$$

$$= \underline{S}^*(\omega)\ \underline{S}(\omega+\nu)$$

- complex ambiguity function

$$\underline{\chi}(t,\nu) = \mathcal{F}^{-1}\{\underline{S}^*(\omega)\ S(\omega+\nu)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underline{S}^*(\omega)\ \underline{S}(\omega+\nu)e^{j\omega t}\,d\omega$$

- ambiguity function

$$|\underline{\chi}(t,\nu)| = \left|\frac{1}{2\pi}\int_{-\infty}^{+\infty} \underline{S}^*(\omega)\ \underline{S}(\omega+\nu)e^{j\omega t}\,d\omega\right|$$

Properties of the Ambiguity Function

1.) maximum value

$$|\underline{\chi}(t, \nu)| \leq |\underline{\chi}(0,0)| = E$$

2.) symmetry

$$|\underline{\chi}(-t, -\nu)| = |\underline{\chi}(t, \nu)|$$

3.) constant volume

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |\underline{\chi}(t, \nu)|^2 \, d\nu \, dt = E^2$$

4.) effect of chirping

Let $\underline{s}(t)$ have the ambiguity function $|\underline{\chi}(t, \nu)|$.

Then the ambiguity function of $\underline{s}(t)\, e^{j \frac{1}{2} k t^2}$ reads $|\underline{\chi}(t, \nu - kt)|$.

Discussion

Compare waveforms of constant, e.g., normalized energy:

- According to 1.) the maximum value of the ambiguity function is constant.

- According to 3.) the volume under the squared ambiguity function is constant.

$\Rightarrow$ One may optimize waveforms in such a way that the ambiguity function is squeezed along one axis, e.g., the delay axis, but then the volume must spread out along the other axis, e.g., the frequency axis.

Cuts Through the Ambiguity Function

1.) $\nu = 0$

$$|\underline{\chi}(t,0)| = \left| \int_{-\infty}^{+\infty} \underline{s}^*(\tau - t)\, \underline{s}(\tau)\, d\tau \right|$$

$$= \left| \int_{-\infty}^{+\infty} \underline{s}^*(\tau')\, \underline{s}(\tau' + t)\, d\tau' \right|$$

$$= \left| \underline{R}_{ss}(t) \right|$$

$$= \left| \mathcal{F}^{-1}\left\{ |\underline{S}(\omega)|^2 \right\} \right|$$

$$= \left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\underline{S}(\omega)|^2 e^{j\omega t}\, d\omega \right|$$

2.) $t = 0$

$$|\underline{\chi}(0,\nu)| = \left| \int_{-\infty}^{+\infty} \underline{s}^*(\tau)\, \underline{s}(\tau)\, e^{-j\nu\tau}\, d\tau \right|$$

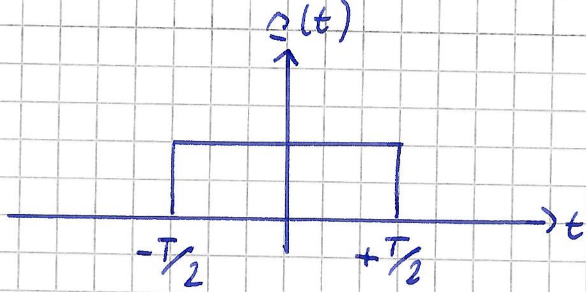
$$= \left| \int_{-\infty}^{+\infty} |\underline{s}(\tau)|^2 e^{-j\nu\tau}\, d\tau \right|$$

$$= \left| \mathcal{F}\left\{ |\underline{s}(\tau)|^2 \right\} \right|$$

Example: Rectangular Pulse

- rectangular pulse

$$\underline{s}(t) = \mathrm{rect}\left(\frac{t}{T}\right)$$



- ambiguity function

$$|\underline{\chi}(t,\nu)| = \left| \int_{-\infty}^{+\infty} \mathrm{rect}\left(\frac{\tau - t}{T}\right) \mathrm{rect}\left(\frac{\tau}{T}\right) e^{-j\nu\tau} \, d\tau \right|$$

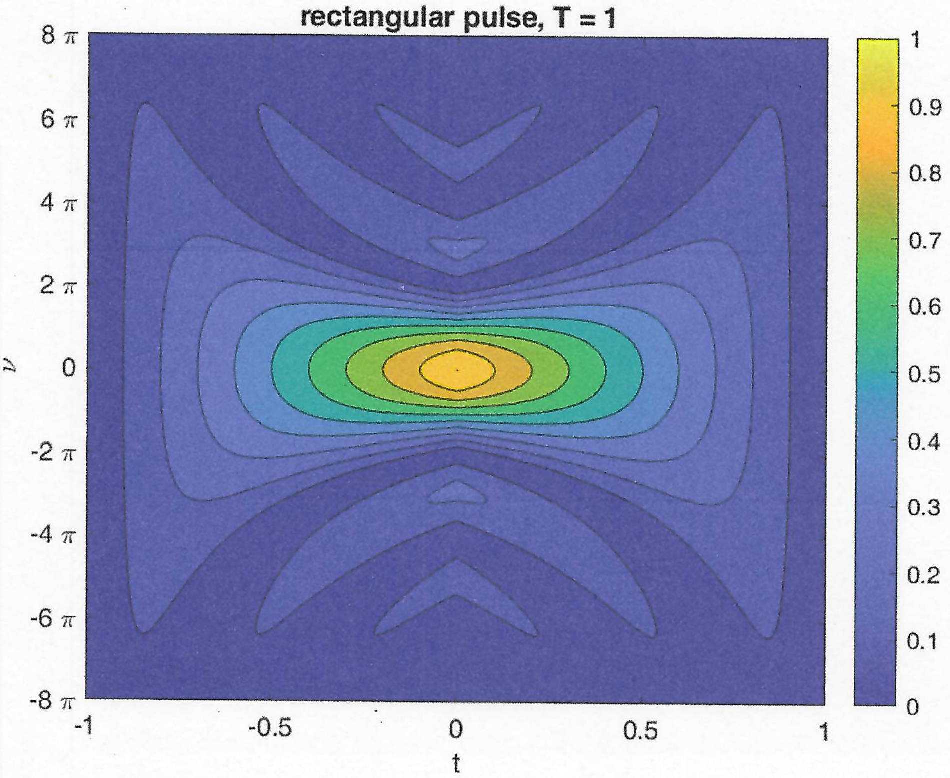
$$= \begin{cases} \left| \int_{-\infty}^{+\infty} \mathrm{rect}\left(\frac{\tau}{T-|t|} - \frac{t}{2}\right) e^{-j\nu\tau} \, d\tau \right| & |t| < T \\ 0 & \text{otherwise} \end{cases}$$

| Fourier transform of $\mathrm{rect}\left(\frac{\tau}{T-|t|} - \frac{t}{2}\right)$

$$= \begin{cases} \left| (T-|t|) \, \mathrm{si}\left(\frac{\nu(T-|t|)}{2}\right) e^{-j\nu\frac{t}{2}} \right| & |t| < T \\ 0 & \text{otherwise} \end{cases}$$

$$= T \, \Lambda\left(\frac{t}{T}\right) \left| \mathrm{si}\left(\frac{\nu(T-|t|)}{2}\right) \right|$$

normalized ambiguity function



Pulse Train

- pulse train

$$\underline{s}(t) = \sum_{k=0}^{K-1} \underline{c}_k \, \underline{s}_p(t - kT)$$

$\underline{s}_p$ → basis pulse

$\underline{c}_k$ → code

- complex ambiguity function

$$\underline{\chi}(t,\nu) = \int_{-\infty}^{+\infty} \sum_m \underline{c}_m^* \, \underline{s}_p^*(\tau - t - mT) \sum_l \underline{c}_l \, \underline{s}_p(\tau - lT) \, e^{-j\nu\tau} \, d\tau$$

$$= \sum_m \sum_l \underline{c}_m^* \underline{c}_l \int_{-\infty}^{+\infty} \underline{s}_p^*(\tau - t - mT) \, \underline{s}_p(\tau - lT) \, e^{-j\nu\tau} \, d\tau$$

$$\Big| \; \tau' = \tau - lT \quad \Rightarrow \quad \tau = \tau' + lT$$

$$= \sum_m \sum_l \underline{c}_m^* \underline{c}_l \int_{-\infty}^{+\infty} \underline{s}_p^*(\tau' - t + (l-m)T) \, \underline{s}_p(\tau') \, e^{-j\nu(\tau' + lT)} \, d\tau'$$

$$\Big| \; k = l - m \quad \Rightarrow \quad m = l - k$$

$$= \sum_k \sum_l \underline{c}_{l-k}^* \underline{c}_l \, e^{-j\nu lT} \underbrace{\int_{-\infty}^{+\infty} \underline{s}_p^*(\tau' - t + kT) \, \underline{s}_p(\tau') \, e^{-j\nu\tau'} \, d\tau'}_{\underline{\chi}_p(t - kT, \nu)}$$

complex ambiguity function of the basis pulse

Special Case: Unmodulated Train of Pulses

- unmodulated pulses

$\underline{c}_k = 1 \quad , k = 0, \ldots, K-1$

- complex ambiguity function

$$\underline{\chi}(t,\nu) = \sum_{k=-K+1}^{-1} \sum_{l=0}^{K-1+k} e^{-j\nu l T} \underline{\chi}_p(t-kT,\nu) \qquad (k \text{ negative})$$

$$+ \sum_{l=0}^{K-1} e^{-j\nu l T} \underline{\chi}_p(t,\nu) \qquad (k=0)$$

$$+ \sum_{k=1}^{K-1} \sum_{l=k}^{K-1} e^{-j\nu l T} \underline{\chi}_p(t-kT,\nu) \qquad (k \text{ positive})$$

$$= \sum_{k=-K+1}^{-1} \underline{\chi}_p(t-kT,\nu) \sum_{l=0}^{K-1-|k|} e^{-j\nu l T}$$

$$+ \underline{\chi}_p(t,\nu) \sum_{l=0}^{K-1} e^{-j\nu l T}$$

$$+ \sum_{k=1}^{K-1} \underline{\chi}_p(t-kT,\nu)\, e^{-j\nu k T} \sum_{l=0}^{K-1-|k|} e^{-j\nu l T}$$

- geometric series

$$\sum_{\ell=0}^{K-1-|k|} e^{-j\nu \ell T} = \frac{1-e^{-j\nu (K-|k|)T}}{1-e^{-j\nu T}}$$

$$= \frac{e^{-j\nu \frac{K-|k|}{2}T}}{e^{-j\nu T/2}} \frac{e^{j\nu \frac{K-|k|}{2}T} - e^{-j\nu \frac{K-|k|}{2}T}}{e^{j\nu T/2} - e^{-j\nu T/2}}$$

$$= (K-|k|)\, e^{-j\nu \frac{K-1-|k|}{2}T} \underbrace{\frac{\sin\left(\nu \frac{K-|k|}{2} T\right)}{(K-|k|)\sin\left(\nu T/2\right)}}_{di_{K-|k|}(\nu T)}$$

- Dirichlet function

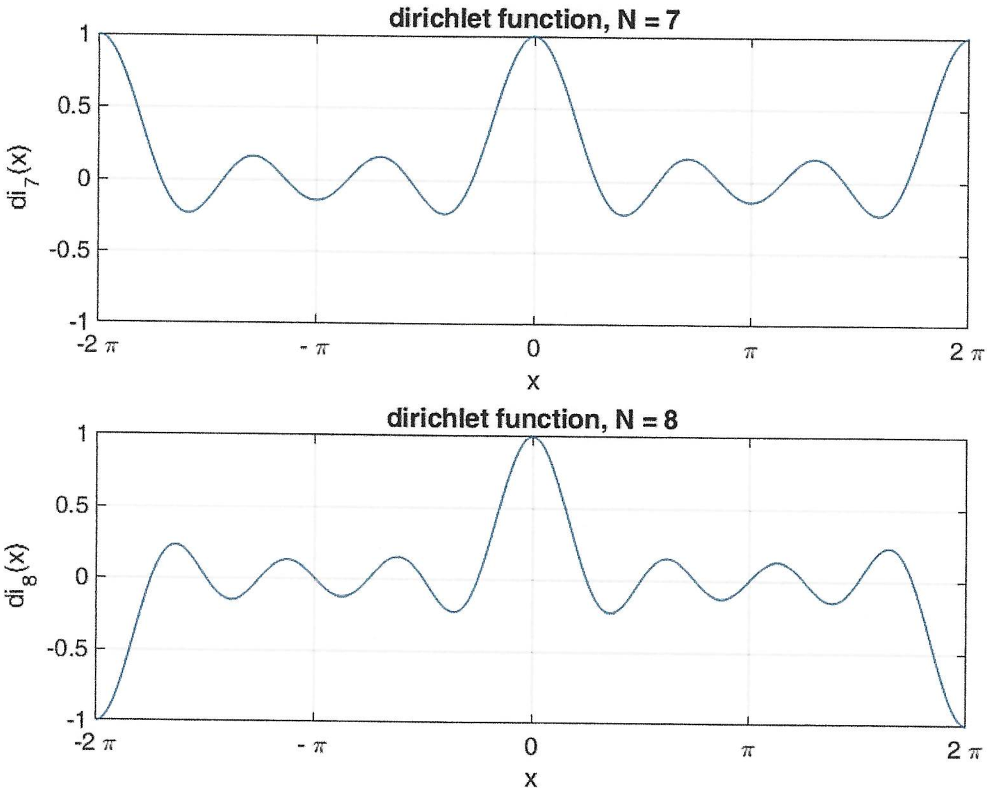
$$di_N(x) = \frac{\sin\left(\frac{Nx}{2}\right)}{N\sin\left(\frac{x}{2}\right)}$$

properties:

1.) $di_N(x) = di_N(-x)$

2.) $di_N(0) = \lim\limits_{N\to\infty} \frac{\sin\left(\frac{Nx}{2}\right)}{N\sin\left(\frac{x}{2}\right)} = \lim\limits_{x\to 0} \frac{\frac{N}{2}\cos\left(\frac{Nx}{2}\right)}{\frac{N}{2}\cos\left(\frac{x}{2}\right)} = 1$

3.) $di_N(x+2\pi) = (-1)^N di_N(x)$


dirichlet function, N = 7
1
0.5
0
-0.5
-1
$di_7(x)$
-2 π
- π
0
π
2 π
x
dirichlet function, N = 8
1
0.5
0
-0.5
-1
$di_8(x)$
-2 π
- π
0
π
2 π
x

- ambiguity function

$$|\underline{\chi}(t,\nu)|$$

$$= \Bigg| \sum_{k=-K+1}^{-1} \underline{\chi}_p(t-kT,\nu)\,(K-|k|)\,e^{-j\nu\frac{K-1-|k|}{2}T}\,\mathrm{di}_{K-|k|}(\nu T)$$

$$+ \underline{\chi}_p(t,\nu)\,K\,e^{-j\nu\frac{K-1}{2}T}\,\mathrm{di}_K(\nu T)$$

$$+ \sum_{k=1}^{K-1} \underline{\chi}_p(t-kT,\nu)\,(K-|k|)\,e^{-j\nu\frac{K-1-|k|+2k}{2}}\,\mathrm{di}_{K-|k|}(\nu T)\Bigg|$$

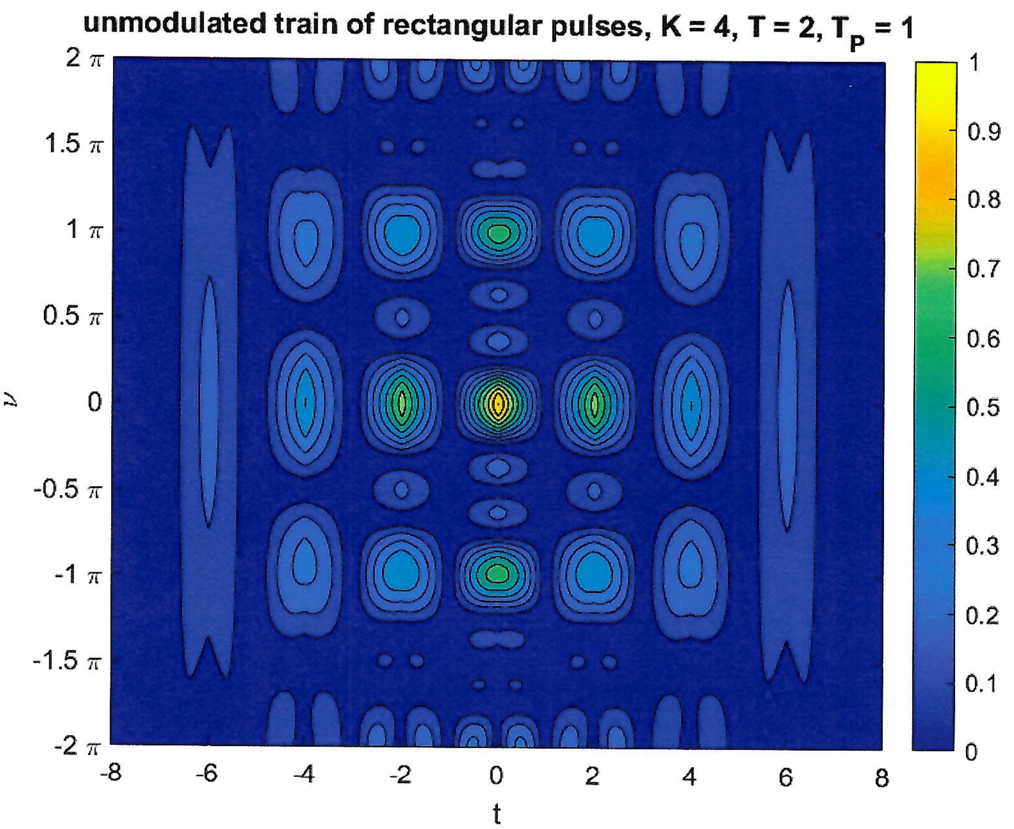
$$= \Bigg| \sum_{k=-K+1}^{K-1} \underline{\chi}_p(t-kT,\nu)\,(K-|k|)\,e^{-j\nu\frac{K-1+k}{2}}\,\mathrm{di}_{K-|k|}(\nu T)\Bigg|$$

• non-overlapping complex ambiguity function of the basic pulse, i.e., time limited basic pulse

$$\underline{\chi}_p(\Delta t, \nu) = \begin{cases} \underline{\chi}_p(\Delta t, \nu) & -T/2 < \Delta t < T/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow |\underline{\chi}(\Delta t, \nu)| = \sum_{k=-K+1}^{K-1} |\underline{\chi}_p(\Delta t - kT, \nu)| \, (K-|k|) \, |di_{K-|k|}(\nu T)|$$

normalized ambiguity function



compare pulse radar
range and velocity ambiguity