Extended Kalman Filter

Non Linear Dynamical System Model

- for simplicity
  - time invariant model
  - zero mean noise

- state transition equation

$$\underbrace{\begin{pmatrix} x_1(n+1) \\ \vdots \\ x_K(n+1) \end{pmatrix}}_{x(n+1)} = \underbrace{\begin{pmatrix} f_1^{(n)}(x(n)) \\ \vdots \\ f_K^{(n)}(x(n)) \end{pmatrix}}_{f^{(n)}(x(n))} + u(n)$$

$\hookrightarrow$ zero mean state noise, correlation matrix $R_{uu}(n) = E\{u(n) \cdot u^T(n)\}$

- observation equation

$$\underbrace{\begin{pmatrix} y_1(n) \\ \vdots \\ y_L(n) \end{pmatrix}}_{y(n)} = \underbrace{\begin{pmatrix} g_1^{(n)}(x(n)) \\ \vdots \\ g_L^{(n)}(x(n)) \end{pmatrix}}_{g^{(n)}(x(n))} + v(n)$$

$\hookrightarrow$ zero mean observation noise, correlation matrix $R_{vv}(n) = E\{v(n) \cdot v^T(n)\}$

Linear Approximations

$$F(n) = \begin{pmatrix} \left.\frac{\partial f_1^{(n)}}{\partial x_1}\right|_{\hat{x}(n|n)} & \cdots & \left.\frac{\partial f_1^{(n)}}{\partial x_K}\right|_{\hat{x}(n|n)} \\ \vdots & & \vdots \\ \left.\frac{\partial f_K^{(n)}}{\partial x_1}\right|_{\hat{x}(n|n)} & \cdots & \left.\frac{\partial f_K^{(n)}}{\partial x_K}\right|_{\hat{x}(n|n)} \end{pmatrix}$$

$$G(n) = \begin{pmatrix} \left.\frac{\partial g_1^{(n)}}{\partial x_1}\right|_{\hat{x}(n|n-1)} & \cdots & \left.\frac{\partial g_1^{(n)}}{\partial x_K}\right|_{\hat{x}(n|n-1)} \\ \vdots & & \vdots \\ \left.\frac{\partial g_L^{(n)}}{\partial x_1}\right|_{\hat{x}(n|n-1)} & \cdots & \left.\frac{\partial g_L^{(n)}}{\partial x_K}\right|_{\hat{x}(n|n-1)} \end{pmatrix}$$

Extended Kalman Filter

idea: use the linear approximations when applying the Kalman filter to a non linear dynamical system model

initialize $\hat{x}(0\|-1)$ and $R_{xx}(0\|-1)$
for $n = 0, \ldots, N$
estimation step $K(n) = R_{xx}(n\|n-1) \cdot G^T(n) \cdot \left(G(n) \cdot R_{xx}(n\|n-1) \cdot G^T(n) + R_{vv}\right)^{-1}$ $\hat{x}(n\|n) = \hat{x}(n\|n-1) + K(n) \cdot \left(y(n) - g^{(n)}(\hat{x}(n\|n-1))\right)$ $R_{xx}(n\|n) = \left(I - K(n) \cdot G(n)\right) \cdot R_{xx}(n\|n-1)$
prediction step $\hat{x}(n+1\|n) = f^{(n)}(\hat{x}(n\|n))$ $R_{xx}(n+1\|n) = F(n) \cdot R_{xx}(n\|n) \cdot F^T(n) + R_{uu}$

heuristic approach, does not satisfy any optimality criteria

Example: Time of Arrival (TOA) Tracking

- non linear observation equation

$$r_\ell(n) = g_\ell^{(n)}\left(x(n), y(n)\right)$$
$$= \sqrt{(x(n)-x_\ell)^2 + (y(n)-y_\ell)^2}$$

- linear approximation

$$\left.\frac{\partial g_\ell^{(n)}}{\partial x}\right|_{x(n), y(n)} = \frac{x(n)-x_\ell}{\sqrt{(x(n)-x_\ell)^2 + (y(n)-y_\ell)^2}}$$

$$\left.\frac{\partial g_\ell^{(n)}}{\partial y}\right|_{x(n), y(n)} = \frac{y(n)-y_\ell}{\sqrt{(x(n)-x_\ell)^2 + (y(n)-y_\ell)^2}}$$