Kalman Filter

Recursive Estimator

- goal: determine an estimate $\hat{x}(n|n)$ of the state vector $x(n)$ based on the current and all past observations $y(0), \ldots, y(n)$
  ⇒ real time approach

- computationally efficient recursive algorithm:

  - determine the estimate $\hat{x}(n|n)$ based on the prediction $\hat{x}(n|n-1)$ and the current observation $y(n)$
    ⇒ data fusion

  - determine the next prediction $\hat{x}(n+1|n)$

- linear dynamical system model
  ⇒ Kalman filter

Estimation Step

- innovation $\quad y(n) - \underbrace{G(n) \cdot \hat{x}(n|n-1)}$

  LMMSE estimate of $y(n)$ based on $\hat{x}(n|n-1)$, see observation equation

- correlation matrix of the innovation

$$E\{(y(n) - G(n) \cdot \hat{x}(n|n-1)) \cdot (y(n) - G(n) \cdot \hat{x}(n|n-1))^T\}$$

| observation equation

$$= E\{(G(n) \cdot x(n) + v(n) - G(n) \cdot \hat{x}(n|n-1))$$

$$\cdot (G(n) \cdot x(n) + v(n) - G(n) \cdot \hat{x}(n|n-1))^T\}$$

| $x(n)$ and $v(n)$ are uncorrelated
| $\hat{x}(n|n-1)$ and $v(n)$ are uncorrelated

$$= G(n) \cdot \underbrace{E\{(x(n) - \hat{x}(n|n-1)) \cdot (x(n) - \hat{x}(n|n-1))^T\}}_{R_{xx}(n|n-1)} \cdot G^T(n)$$

correlation matrix of the prediction

$$+ \underbrace{E\{v(n) \cdot v^T(n)\}}_{R_{vv}(n)}$$

$$= G(n) \cdot R_{xx}(n|n-1) \cdot G^T(n) + R_{vv}(n)$$

• cross-correlation matrix of $x(n)$ and the innovation

$$E\{x(n)\cdot(y(n)-G(n)\cdot\hat{x}(n|n-1))^T\}$$

| observation equation

$$=E\{x(n)\cdot(G(n)\cdot x(n)+v(n)-G(n)\cdot\hat{x}(n|n-1))^T\}$$

| $x(n)$ and $v(n)$ are uncorrelated

$$=E\{x(n)\cdot(x(n)-\hat{x}(n|n-1))^T\}\cdot G^T(n)$$

| orthogonality principle

$$=\underbrace{E\{(x(n)-\hat{x}(n|n-1))\cdot(x(n)-\hat{x}(n|n-1))^T\}}_{R_{xx}(n|n-1)}\cdot G^T(n)$$

$$=R_{xx}(n|n-1)\cdot G^T(n)$$

- Kalman gain matrix

$$K(n) = \underbrace{R_{xx}(n|n-1) \cdot G^T(n)}_{\text{cross-correlation matrix of } x(n) \text{ and the innovation}} \cdot \underbrace{\left(G(n) \cdot R_{xx}(n|n-1) \cdot G^T(n) + R_{vv}(n) \right)^{-1}}_{\text{correlation matrix of the innovation}}$$

- LMMSE estimate of $x(n)$ based on the prediction $\hat{x}(n|n-1)$ and $y(n)$

$$\hat{x}(n|n) = \underbrace{\hat{x}(n|n-1)}_{\text{LMMSE estimate of } x \text{ based on } \hat{x}(n|n-1)} + K(n) \cdot \underbrace{\left(y(n) - G(n) \cdot \hat{x}(n|n-1) \right)}_{\text{innovation}}$$

$$= (\underbrace{I}_{\text{identity matrix}} - K(n) \cdot G(n)) \cdot \hat{x}(n|n-1) + K(n) \cdot y(n)$$

Estimation Error

correlation matrix of the estimation error $\hat{x}(n) - x(n)$

$$R_{xx}(n|n) = \underbrace{R_{xx}(n|n-1)}_{\substack{\text{correlation matrix} \\ \text{of the estimation} \\ \text{error } \hat{x}(n|n-1) - x(n)}} + \underbrace{K(n)}_{\substack{\text{Kalman} \\ \text{gain matrix}}} \cdot \underbrace{G(n) \cdot R_{xx}(n|n-1)}_{\substack{\text{cross-correlation matrix of the} \\ \text{innovation and } x(n) \\ = \text{transpose of the} \\ \text{cross-correlation matrix} \\ \text{of } x(n) \text{ and the innovation}}}$$

$$= \left(I + K(n) \cdot G(n) \right) \cdot R_{xx}(n|n-1)$$

Prediction Step

LMMSE estimate of $x(n+1)$ based on $\hat{x}(n|n)$, see state transition equation

$$\hat{x}(n+1|n) = F(n) \cdot \hat{x}(n|n)$$

Prediction Error

correlation matrix of the prediction error $\hat{x}(n+1|n) - x(n+1)$

$$R_{xx}(n+1|n) = E\left\{\left(\hat{x}(n+1|n) - x(n+1)\right)\cdot\left(\hat{x}(n+1|n) - x(n+1)\right)^T\right\}$$

| prediction step

$$= E\left\{\left(F(n)\cdot\hat{x}(n|n) - x(n+1)\right)\cdot\left(F(n)\cdot\hat{x}(n|n) - x(n+1)\right)^T\right\}$$

| state transition equation

$$= E\left\{\left(F(n)\cdot\hat{x}(n|n) - F(n)\cdot x(n) - u(n)\right)\cdot\left(F(n)\cdot\hat{x}(n|n) - F(n)\cdot x(n) - u(n)\right)^T\right\}$$

| $x(n)$ and $u(n)$ are uncorrelated
| $\hat{x}(n|n)$ and $u(n)$ are uncorrelated

$$= F(n)\cdot\underbrace{E\left\{\left(\hat{x}(n|n) - x(n)\right)\cdot\left(\hat{x}(n|n) - x(n)\right)^T\right\}}_{R_{xx}(n|n)}\cdot F^T(n)$$

$$+ \underbrace{E\left\{u(n)\cdot u^T(n)\right\}}_{R_{uu}(n)}$$

$$= F(n)\cdot R_{xx}(n|n)\cdot F^T(n) + R_{uu}(n)$$

Summary

initialize $\hat{x}(0|-1)$ and $R_{xx}(0|-1)$

for $n = 0, \dots, N$

estimation step

$$K(n) = R_{xx}(n|n-1) \cdot G^T(n) \cdot \left(G(n) \cdot R_{xx}(n|n-1) \cdot G^T(n) + R_{vv}(n) \right)^{-1}$$

$$\hat{x}(n|n) = \hat{x}(n|n-1) + K(n) \cdot \left(y(n) - G(n) \cdot \hat{x}(n|n-1) \right)$$

$$R_{xx}(n|n) = \left(I - K(n) \cdot G(n) \right) \cdot R_{xx}(n|n-1)$$

prediction step

$$\hat{x}(n+1|n) = F(n) \cdot \hat{x}(n|n)$$

$$R_{xx}(n+1|n) = F(n) \cdot R_{xx}(n|n) \cdot F^T(n) + R_{uu}(n)$$

Block Diagramm of the Kalman Filter