Estimation Theory

Linear Minimum Mean Square Error (LMMSE) Estimator

- Compute an estimate $\hat{x}$ of the parameter vector x based on the observation vector y!
- use a linear estimator

$$\hat{x} = W \cdot y$$

- mean square error

$$E\{\|\hat{x}-x\|^2\} = E\{\operatorname{trace}((\hat{x}-x)\cdot(\hat{x}-x)^T)\}$$

$$= E\{\operatorname{trace}((W\cdot y - x)\cdot(W\cdot y - x)^T)\}$$

$$= E\{\operatorname{trace}(W\cdot y\cdot y^T\cdot W - W\cdot y\cdot x^T - x\cdot y^T\cdot W^T + x\cdot x^T)\}$$

$$= \operatorname{trace}(W\cdot \underbrace{E\{y\cdot y^T\}}_{R_{yy} = R_{yy}^T}\cdot W^T - W\cdot \underbrace{E\{y^T\cdot x\}}_{R_{yx} = R_{xy}^T}$$

$$- \underbrace{E\{x\cdot y^T\}}_{R_{xy}}\cdot W^T + \underbrace{E\{x\cdot x^T\}}_{R_{xx} = R_{xx}^T}$$

| completing the squares

$$= \operatorname{trace}((W\cdot R_{yy} - R_{xy})\cdot R_{yy}^{-1}\cdot(W\cdot R_{yy} - R_{xy})^T$$

$$- R_{xy}\cdot R_{yy}^{-1}\cdot R_{xy}^T + R_{xx})$$

- as R_{yy} is a positive semidefinite matrix the mean square error is minimized by

$$W\cdot R_{yy} - R_{xy} = 0$$

$$W = R_{xy}\cdot R_{yy}^{-1} \quad \text{(Wiener estimator)}$$

$$\Rightarrow \hat{x} = R_{xy}\cdot R_{yy}^{-1}\cdot y$$

Residual Error

correlation matrix of the residual error vector

$$R_{\hat{x}-x,\hat{x}-x} = E\{(\hat{x}-x)\cdot(\hat{x}-x)^T\}$$

$$= E\{(W\cdot y - x)\cdot(W\cdot y - x)^T\}$$

$\vdots$ see above

$$= \underbrace{(W\cdot R_{yy} - R_{xy})\cdot R_{yy}^{-1}\cdot(W\cdot R_{yy} - R_{xy})^T}_{0}$$

$$- R_{xy}\cdot R_{yy}^{-1}\cdot R_{xy}^T + R_{xx}$$

$$= R_{xx} - R_{xy}\cdot R_{yy}^{-1}\cdot R_{xy}^T$$

Orthogonality Principle

- the residual error vector $\hat{x}-x$ and the observation vector y are uncorrelated

$$
\begin{aligned}
E\{(\hat{x}-x)\cdot y^T\} &= E\{(W\cdot y - x)\cdot y^T\} \\
&= E\{W\cdot y\cdot y^T - x\cdot y^T\} \\
&= W\cdot \underbrace{E\{y\cdot y^T\}}_{R_{yy}} - \underbrace{E\{x\cdot y^T\}}_{R_{xy}} \\
&= R_{xy}\cdot R_{yy}^{-1}\cdot R_{yy} - R_{xy} \\
&= 0
\end{aligned}
$$

- the residual error vector $\hat{x}-x$ and the estimate $\hat{x}$ are uncorrelated

$$
\begin{aligned}
E\{(\hat{x}-x)\cdot \hat{x}^T\} &= E\{(\hat{x}-x)\cdot (W\cdot y)^T\} \\
&= E\{(\hat{x}-x)\cdot y^T\cdot W\} \\
&= \underbrace{E\{(\hat{x}-x)\cdot y^T\}}_{0}\cdot W \\
&= 0
\end{aligned}
$$

Data Fusion

- Compute an estimate $\hat{x}$ of the parameter vector x based on the observation vectors y_1 and y_2.

- correlation matrices

$$R_{x,y_1y_2} = E\{x \cdot (y_1^T, y_2^T)\}$$

$$= E\{(x \cdot y_1^T, x \cdot y_2^T)\}$$

$$= (R_{xy_1}, R_{xy_2})$$

$$R_{y_1y_2,y_1y_2} = E\left\{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \cdot (y_1^T, y_2^T)\right\}$$

$$= E\left\{\begin{pmatrix} y_1 \cdot y_1^T & y_1 \cdot y_2^T \\ y_2 \cdot y_1^T & y_2 \cdot y_2^T \end{pmatrix}\right\}$$

$$= \begin{pmatrix} R_{y_1y_1} & R_{y_1y_2} \\ R_{y_2y_1} & R_{y_2y_2} \end{pmatrix}$$

• block matrix inversion

$$R_{y_1y_2,y_1y_2}^{-1} = \begin{pmatrix} R_{y_1y_1}^{-1} + R_{y_1y_1}^{-1} \cdot R_{y_1y_2} \cdot \left(R_{y_2y_2} - R_{y_2y_1} \cdot R_{y_1y_1}^{-1} \cdot R_{y_1y_2}\right)^{-1} \cdot R_{y_2y_1} \cdot R_{y_1y_1}^{-1} & -R_{y_1y_1}^{-1} \cdot R_{y_1y_2} \cdot \left(R_{y_2y_2} - R_{y_2y_1} \cdot R_{y_1y_1}^{-1} \cdot R_{y_1y_2}\right)^{-1} \\ -\left(R_{y_2y_2} - R_{y_2y_1} \cdot R_{y_1y_1}^{-1} \cdot R_{y_1y_2}\right)^{-1} \cdot R_{y_2y_1} \cdot R_{y_1y_1}^{-1} & \left(R_{y_2y_2} - R_{y_2y_1} \cdot R_{y_1y_1}^{-1} \cdot R_{y_1y_2}\right)^{-1} \end{pmatrix}$$

• LMMSE estimator

$$\hat{x} = R_{x,y_1y_2} \cdot R_{y_1y_2,y_1y_2}^{-1} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \left(\begin{array}{l} R_{xy_1} \cdot R_{y_1y_1}^{-1} - \left(R_{xy_2} - R_{xy_1} \cdot R_{y_1y_1}^{-1} \cdot R_{y_1y_2}\right) \cdot \left(R_{y_2y_2} - R_{y_2y_1} \cdot R_{y_1y_1}^{-1} \cdot R_{y_1y_2}\right)^{-1} \cdot R_{y_2y_1} \cdot R_{y_1y_1}^{-1} \\ \left(R_{xy_2} - R_{xy_1} \cdot R_{y_1y_1}^{-1} \cdot R_{y_1y_2}\right) \cdot \left(R_{y_2y_2} - R_{y_2y_1} \cdot R_{y_1y_1}^{-1} \cdot R_{y_1y_2}\right)^{-1} \end{array} \right) \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \underbrace{R_{xy_1} \cdot R_{y_1y_1}^{-1} \cdot y_1}_{\text{LMMSE estimate of } x \text{ based on } y_1} + \underbrace{\underbrace{\left(R_{xy_2} - R_{xy_1} \cdot R_{y_1y_1}^{-1} \cdot R_{y_1y_2}\right)}_{\text{cross-correlation matrix of } x \text{ and the innovation}} \cdot \underbrace{\left(R_{y_2y_2} - R_{y_2y_1} \cdot R_{y_1y_1}^{-1} \cdot R_{y_1y_2}\right)}_{\text{correlation matrix of the innovation}}^{-1}}_{K \text{ Kalman gain matrix}} \cdot \underbrace{\left(y_2 - \underbrace{R_{y_2y_1} \cdot R_{y_1y_1}^{-1} \cdot y_1}_{\text{LMMSE estimate of } y_2 \text{ based on } y_1}\right)}_{\text{innovation}}$$

Residual Error

correlation matrix of the residual error vector

$$R_{\hat{x}-x,\hat{x}-x} = R_{xx} - R_{x,y_1y_2} \cdot R^{-1}_{y_1y_2,y_1y_2} \cdot R^{T}_{x,y_1y_2}$$

$$= R_{xx} - \Big(R_{xy_1} \cdot R^{-1}_{y_1y_1} - (R_{xy_2} - R_{xy_1} \cdot R^{-1}_{y_1y_1} \cdot R_{y_1y_2}) \cdot (R_{y_2y_2} - R_{y_2y_1} \cdot R^{-1}_{y_1y_1} \cdot R_{y_1y_2})^{-1} \cdot R_{y_2y_1} \cdot R^{-1}_{y_1y_1} \,,$$

$$(R_{xy_2} - R_{xy_1} \cdot R^{-1}_{y_1y_1} \cdot R_{y_1y_2}) \cdot (R_{y_2y_2} - R_{y_2y_1} \cdot R^{-1}_{y_1y_1} \cdot R_{y_1y_2})^{-1} \Big) \cdot \begin{pmatrix} R^T_{xy_1} \\ R^T_{xy_2} \end{pmatrix}$$

$$= \underbrace{R_{xx} - R_{xy_1} \cdot R^{-1}_{y_1y_1} \cdot R^T_{xy_1}}$$

correlation matrix of the residual error vector
of the LMMSE estimate based on y_1

$$+ \underbrace{(R_{xy_2} - R_{xy_1} \cdot R^{-1}_{y_1y_1} \cdot R_{y_1y_2}) \cdot (R_{y_2y_2} - R_{y_2y_1} \cdot R^{-1}_{y_1y_1} \cdot R_{y_1y_2})^{-1}}_{K} \cdot \underbrace{(R_{y_2x} - R_{y_2y_1} \cdot R^{-1}_{y_1y_1} \cdot R_{y_1x})}$$

Kalman gain matrix

cross-correlation matrix
of the innovation and x