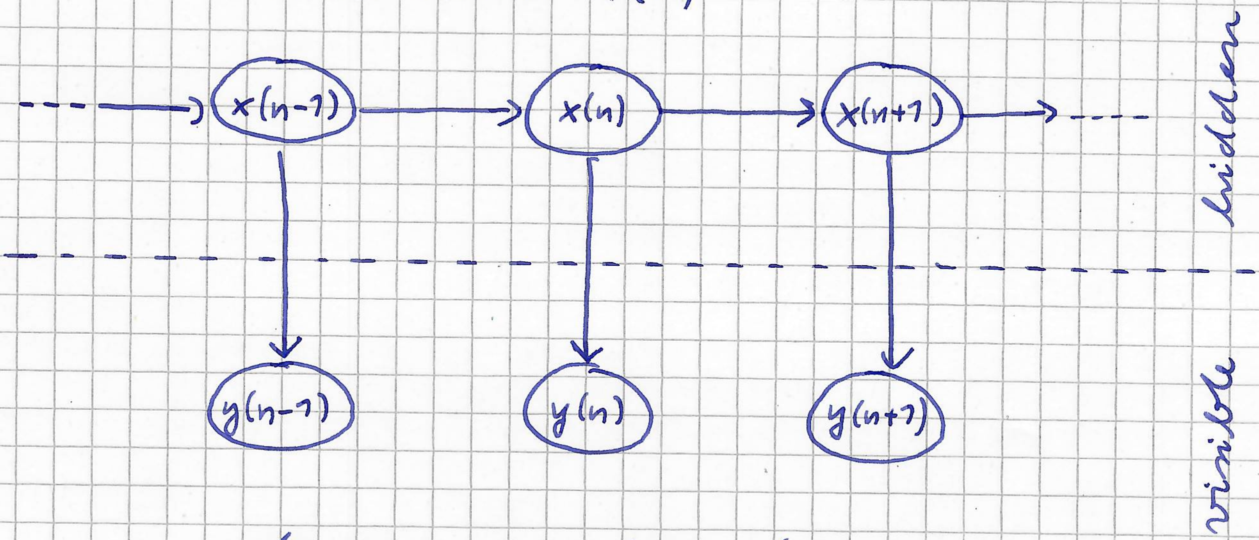
# Bayesian Filter Approach

# Statistical Dependencies in Hidden Markov Models

state vector $x(n)$




observation vector $y(n)$

- conditional probability density function (pdf) of the state $x(n)$ depends only on the previous state $x(n-1)$:

$$p(x(n) \mid x(0), \ldots, x(n-1), y(0), \ldots, y(n-1))$$

$$= p(x(n) \mid x(n-1))$$

- conditional probability density function (pdf) of the observation $y(n)$ depends only on the current state $x(n)$:

$$p(y(n) \mid x(0), \ldots, x(n), y(0), \ldots, y(n-1))$$

$$= p(y(n) \mid x(n))$$

# State Estimation Problem

- The hidden Markov model, i.e., the state transition pdf $p(x(n) | x(n-1))$, the observation pdf $p(y(n) | x(n))$, and the initial state pdf $p(x(0))$ shall be known.

- Estimate the current state $x(n)$ based on all previous and the current observations, i.e., determine the posterior $p(x(n) | y(0), \ldots, y(n))$.

# Estimation Step

- the prior $p(x(n)|y(0),\ldots,y(n-1))$ shall be known

- now also consider the current observation $y(n)$:

$$\underbrace{p(x(n)|y(0),\ldots,y(n-1),y(n))}_{\text{posterior}}$$

$$= \frac{p(x(n),y(n)|y(0),\ldots,y(n-1))}{p(y(n)|y(0),\ldots,y(n-1))}$$

$$= \frac{p(y(n)|x(n),y(0),\ldots,y(n-1)) \cdot p(x(n)|y(0),\ldots,y(n-1))}{p(y(n)|y(0),\ldots,y(n-1))}$$

| Markov property

$$= \frac{\overbrace{p(y(n)|x(n))}^{\text{observation pdf, likelihood function}} \cdot \overbrace{p(y(n)|y(0),\ldots,y(n-1))}^{\text{prior, from prediction step}}}{\underbrace{p(y(n)|y(0),\ldots,y(n-1))}_{\text{evidence}}}$$

- evidence does not depend on $x(n)$
  $\Rightarrow$ normalizing constant in the denominator

- total probability

$$1 = \int p(x(n) \mid y(0), \dots, y(n)) \, dx(n)$$

$$= \frac{\int p(y(n) \mid x(n)) \cdot p(x(n) \mid y(0), \dots, y(n-1)) \, dx(n)}{p(y(n) \mid y(0), \dots, y(n-1))}$$

$$\Rightarrow p(y(n) \mid y(0), \dots, y(n-1))$$

$$= \int p(y(n) \mid x(n)) \cdot p(x(n) \mid y(0), \dots, y(n-1)) \, dx(n)$$

## Prediction Step

- the posterior $p(x(n) | y(0), \dots, y(n))$ shall be known

- now consider the next state $x(n+1)$:

$$p(x(n+1), x(n) | y(0), \dots, y(n))$$

$$= p(x(n+1) | x(n), y(0), \dots, y(n)) \cdot p(x(n) | y(0), \dots, y(n))$$

| Markov property

$$= p(x(n+1) | x(n)) \cdot p(x(n) | y(0), \dots, y(n))$$

$\Rightarrow$ Chapman-Kolmogoroff equation (computed as marginal pdf)

$$\underbrace{p(x(n+1) | y(0), \dots, y(n))}_{\text{prior}}$$

$$= \int \underbrace{p(x(n+1) | x(n))}_{\text{state transition pdf}} \cdot \underbrace{p(x(n) | y(0), \dots, y(n))}_{\text{posterior from estimation step}} \, dx(n)$$

# Implementation Challenges

Bayesian filter approach requires computations with pdfs (functions!).

- discretize the pdfs
  => grid-based Bayesian filter
- represent pdfs by samples drawn from the pdfs
  => particle filter
- consider only Gaussian pdfs
  => can be fully described by means and covariances
  => Kalman filter