Radio Navigation and Radar

6. Exercise

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1. Problem

Let $\underline{x}$ be a complex Gaussian random variable with nonzero mean $\underline{A}$ and a variance σ^2, i.e., both real- and imaginary parts have the variance $\sigma^2/2$. The probability density function of $\underline{x}$ reads

$$\mathrm{p}_{\underline{\mathrm{x}}}(\underline{x}) = \frac{1}{\pi\sigma^2}\,\mathrm{e}^{-\frac{|\underline{x}-\underline{A}|^2}{\sigma^2}}\,.$$

In the following some related random variables and their probability density functions shall be studied.

a) The magnitude x and the argument φ are two functions of the two random variables real part $\mathrm{Re}(\underline{x})$ and imaginary part $\mathrm{Im}(\underline{x})$ of $\underline{x}$. Determine the joint probability density function $\mathrm{p}_{\mathrm{x},\varphi}(x,\varphi)$.

b) Show that the magnitude

$$x = |\underline{x}|$$

is a Ricean random variable with the probability density function

$$\mathrm{p}_{\mathrm{x}}(x) = \frac{2x}{\sigma^2}\,\mathrm{e}^{-\frac{x^2+|\underline{A}|^2}{\sigma^2}}\,\mathrm{I}_0\left(\frac{2x\,|\underline{A}|}{\sigma^2}\right).$$

c) Determine the probability density function $\mathrm{p}_{\mathrm{y}}(y)$ of

$$y = |x|^2 = x^2.$$

d) Give an expression for the cumulative distribution function $\Pr\{y < \eta\}$ using Macum's Q function

$$\mathrm{Q}(a,b) = \int_b^\infty x\,\mathrm{e}^{-\frac{x^2+a^2}{2}}\,\mathrm{I}_0(ax)\,\mathrm{d}x.$$

2. Problem

In the following target detection based on an observed complex received vector $\underline{\mathbf{e}}$ with L samples is considered. In the case $\mathcal{H}_0$ of no target being present the received vector consists of noise only

$$\underline{\mathbf{e}} = \underline{\mathbf{n}}.$$

In the case $\mathcal{H}_1$ of a target being present the received vector consist of a signal with additive noise

$$\underline{\mathbf{e}} = \underline{\mathbf{s}} + \underline{\mathbf{n}}.$$

In both cases the noise $\underline{\mathbf{n}}$ is assumed to be multivariate zero mean white Gaussian noise with the probability density function

$$\mathrm{p}(\underline{\mathbf{n}}) = \frac{1}{\left(\pi\sigma^2\right)^L}\,\mathrm{e}^{-\frac{\|\underline{\mathbf{n}}\|^2}{\sigma^2}}.$$

a) First the signal $\underline{\mathbf{s}}$ is assumed to be completely known. Compute the likelihood ratio and design a target detector based on the most efficient test.

b) Next the case of an unknown common phase φ of all samples is considered

$$\underline{\mathbf{e}} = \underline{\mathbf{s}}\,\mathrm{e}^{\mathrm{j}\varphi} + \underline{\mathbf{n}}.$$

Now the probability density function of the received vector $\underline{\mathbf{e}}$ can be computed as the marginal of

$$\mathrm{p}(\underline{\mathbf{e}}|\varphi,\mathcal{H}_1)\,\mathrm{p}(\varphi) = \mathrm{p}\Big(\underline{\mathbf{n}} = \underline{\mathbf{e}} - \underline{\mathbf{s}}\,\mathrm{e}^{\mathrm{j}\varphi}\Big)\,\mathrm{p}(\varphi)$$

using the uniform probability density function

$$\mathrm{p}(\varphi) = \begin{cases} \frac{1}{2\pi} & -\pi < \varphi < +\pi \\ 0 & \text{otherwise} \end{cases}$$

of the phase φ. Compute the likelihood ratio and design a target detector based on the most efficient test.