Radio Navigation and Radar

7. Exercise

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1. Problem

In the following a linear Gaussian model

$$\mathbf{y} = \mathbf{H} \cdot \mathbf{x} + \mathbf{n}$$

is considered. Let

$$\mathbf{R}_{\mathrm{xx}} = \mathrm{E}\left\{\mathbf{x} \cdot \mathbf{x}^{\mathrm{T}}\right\}$$

be the correlation matrix of the zero mean parameter vector $\mathbf{x}$ and let

$$\mathbf{R}_{\mathrm{nn}} = \mathrm{E}\left\{\mathbf{n} \cdot \mathbf{n}^{\mathrm{T}}\right\}$$

be the correlation matrix of the zero mean noise $\mathbf{n}$.

a) Compute the linear minimum mean square error estimate $\hat{\mathbf{x}}_{\mathrm{LMMSE}}$ as a function of $\mathbf{y}$, $\mathbf{H}$, $\mathbf{R}_{\mathrm{xx}}$, and $\mathbf{R}_{\mathrm{nn}}$.

Let the prior probability density function be

$$\mathrm{p}(\mathbf{x}) = \frac{1}{\sqrt{\det(2\pi \mathbf{R}_{\mathrm{xx}})}} \, \mathrm{e}^{-\frac{1}{2}\mathbf{x}^{\mathrm{T}} \cdot \mathbf{R}_{\mathrm{xx}}^{-1} \cdot \mathbf{x}}$$

and let the likelihood function be

$$\mathrm{p}(\mathbf{Y}|\mathbf{x}) = \frac{1}{\sqrt{\det(2\pi \mathbf{R}_{\mathrm{nn}})}} \, \mathrm{e}^{-\frac{1}{2}(\mathbf{y}-\mathbf{H}\cdot\mathbf{x})^{\mathrm{T}} \cdot \mathbf{R}_{\mathrm{nn}}^{-1} \cdot (\mathbf{y}-\mathbf{H}\cdot\mathbf{x})} \, .$$

b) Determine the posterior probability density function $\mathrm{p}(\mathbf{x}|\mathbf{y})$.

c) Show that the maximum a posteriori estimate

$$\hat{\mathbf{x}}_{\mathrm{MAP}} = \arg\max_{\mathbf{x}}\{\mathrm{p}(\mathbf{x}|\mathbf{y})\}$$

is equal to the minimum mean square error estimate $\hat{\mathbf{x}}_{\mathrm{LMMSE}}$.

2. Problem

A linear dynamical system model with the observation equation

$$\mathbf{y}(n) = \mathbf{G}(n) \cdot \mathbf{x}(n) + \mathbf{v}(n)$$

is considered. Let

$$\mathbf{R}_{\mathrm{vv}}(n) = \mathrm{E}\Big\{\mathbf{v}(n) \cdot \mathbf{v}(n)^{\mathrm{T}}\Big\}$$

be the correlation matrix of the zero mean observation noise $\mathbf{v}(n)$ and

$$\mathbf{R}_{\mathrm{xx}}(n|n-1) = \mathrm{E}\Big\{(\hat{\mathbf{x}}(n|n-1) - \mathbf{x}(n)) \cdot (\hat{\mathbf{x}}(n|n-1) - \mathbf{x}(n))^{\mathrm{T}}\Big\}$$

be the correlation matrix of the prediction error $\hat{\mathbf{x}}(n|n-1) - \mathbf{x}(n)$.

a) The estimate is computed as

$$\hat{\mathbf{x}}(n|n) = \hat{\mathbf{x}}(n|n-1) + \mathbf{K}(n) \cdot (\mathbf{y}(n) - \mathbf{G}(n) \cdot \hat{\mathbf{x}}(n|n-1)).$$

Determine $\mathbf{K}(n)$ in such a way that the mean square error $\mathrm{E}\Big\{\|\hat{\mathbf{x}}(n|n) - \mathbf{x}(n)\|^2\Big\}$ is minimized.

b) Show that the correlation matrix

$$\mathbf{R}_{\mathrm{xx}}(n|n) = \mathrm{E}\Big\{(\hat{\mathbf{x}}(n|n) - \mathbf{x}(n)) \cdot (\hat{\mathbf{x}}(n|n) - \mathbf{x}(n))^{\mathrm{T}}\Big\}$$

of the estimation error $\hat{\mathbf{x}}(n|n) - \mathbf{x}(n)$ can be computed as

$$\begin{aligned}\mathbf{R}_{\mathrm{xx}}(n|n) = {} & (\mathbf{I} - \mathbf{K}(n) \cdot \mathbf{G}(n)) \cdot \mathbf{R}_{\mathrm{xx}}(n|n-1) \cdot (\mathbf{I} - \mathbf{K}(n) \cdot \mathbf{G}(n))^{\mathrm{T}} \\ & + \mathbf{K}(n) \cdot \mathbf{R}_{\mathrm{vv}}(n) \cdot \mathbf{K}^{\mathrm{T}}(n).\end{aligned}$$

This numerically more stable form is known as Joseph's form.