Detection Theory
Binary Hypothesis Testing

Based on the observed signal y, the detector has to decide whether there is a target or not.

=) two hypotheses

H₀: null hypothesis, no target present
   =) only noise

H₁: alternative hypothesis, target present
   =) signal plus noise

Two kinds of errors with quite different consequences may occur:

Type 1 error: decide for target present (H₁) even though there is no target present (H₀)
   =) false positive, false alarm

Type 2 error: decide for no target present (H₀) even though there is a target present (H₁)
   =) false negative, missed detection
Decision Regions

- The detector is characterized by its decision regions.

\[ \text{decide for } H_1 \]

- Probability of false alarm

\[ P_F = \Pr \{ y \in R \mid H_0 \} = \int_R p(y \mid H_0) \, dy \]

- Probability of missed detection

\[ P_n = \Pr \{ y \in \overline{R} \mid H_1 \} = \int_{\overline{R}} p(y \mid H_1) \, dy \]

- Probability of detection

\[ P_D = \Pr \{ y \in R \mid H_1 \} = \int_R p(y \mid H_1) \, dy = 1 - P_n \]
Design of Detectors

- find the decision regions
- a compromise between minimizing both probabilities of error is required.
- idea:

Maximise the probability of detection $P_d$ for a given probability of false alarm $P_f$.

≡ find the most efficient test using the method of Lagrangian multipliers
Neyman–Pearson Theorem

\[ P_0 - 2 P_F = \int_R p(y|H_1) \, dy - 2 \int_R p(y|H_0) \, dy \]
\[ = \int_R (p(y|H_1) - 2 \, p(y|H_0)) \, dy \]

- The decision region \( R \) has to contain all observed signals \( y \) with
\[ p(y|H_1) - 2 \, p(y|H_0) > 0 \]
\[ \Rightarrow \frac{p(y|H_1)}{p(y|H_0)} > 2 \]

Neyman–Pearson theorem

The likelihood ratio test is the most efficient test.
Receiver Operating Characteristic (ROC)

Plot the obtained probability of detection $P_d$ as a function of the given probability of false alarm $P_F$. 
Target Detection
Incoherent Radar Receiver

- Typical radar receiver

\[ y(t) \]

\[ \mathbf{r}(t) \xrightarrow{\text{coherent correlator}} \mathbf{e}(t) \xrightarrow{\text{detection}} y \]

\[ \text{incoherent correlator} \]

- Gaussian noise \( n(t) \)

- \( H_0: \) only noise

\( \Rightarrow \) \( x \) is a complex Gaussian random variable with zero mean and variance \( \sigma^2 = N_0 \)

\( \Rightarrow \) \( y \) is an exponential random variable

\[ p(y | H_0) = \begin{cases} \frac{1}{\sigma^2} e^{-\frac{y}{\sigma^2}} & y > 0 \\ 0 & \text{otherwise} \end{cases} \]
\( H_1: \text{signal plus noise} \)

\( \Rightarrow \) \( x \) is a complex Gaussian random variable with mean \( \mu \) and variance \( \sigma^2 = N_0 \)

\( \Rightarrow y \) is a noncentral chi-square random variable with 2 degrees of freedom

\[
\rho(y | H_1) = \begin{cases} 
\frac{1}{\sigma^2} e^{-\frac{\|y\|^2}{2\sigma^2}} I_0 \left( \frac{2 \|y\|^2}{\sigma^2} \right) & y > 0 \\
0 & \text{otherwise}
\end{cases}
\]

I\(_0\): modified Bessel function of the first kind and order 0

\( SNIR \quad \gamma = \frac{\|y\|^2}{\sigma^2} \)

\[
\rho(y | H_1) = \begin{cases} 
\frac{1}{\sigma^2} e^{-\gamma + \frac{\gamma \|y\|^2}{2\sigma^2}} I_0 \left( \frac{2 \sqrt{\gamma} \|y\|^2}{\sigma^2} \right) & y > 0 \\
0 & \text{otherwise}
\end{cases}
\]
Incoherent Detection

* likelihood ratio test

\[
\frac{p(y | H_1)}{p(y | H_0)} = e^{-\nu} I_0 \left( \frac{2\sqrt{\nu}}{\sigma} \right) > \lambda
\]

* \( I_0(.) \) is strictly growing

\( \Rightarrow \) decide for \( H_1 \) if

\[ y > \nu \]

with a properly chosen threshold \( \nu \)
\[ G^2 = 4 \]
\[ \gamma = 2 \]
Probability of False Alarm

- probability of false alarm

\[ P_f = \int_{-\infty}^{\infty} p(y \mid H_0) \, dy \]
\[ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi \sigma^2}} e^{-\frac{y^2}{\sigma^2}} \, dy = \left[ -e^{-\frac{y^2}{\sigma^2}} \right]_{-\infty}^{\infty} \]
\[ = e^{-\frac{\sigma^2}{\sigma^2}} \]
\[ = e^{-1} \]

- constant false alarm rate (CFAR)

\[ \eta = -\sigma^2 \ln(P_f) \]

adjust the threshold \( \eta \) to the noise variance \( \sigma^2 \)
Probability of Detection

Probability of detection

\[ P_D = \int p(y | H_1) \, dy \]

\[ = \int_0^\infty \frac{n^2}{6^2} e^{-\left(\frac{y + \frac{y^2}{2}}{6}\right)} I_0\left(\frac{2\sqrt{xy}}{6}\right) \, dy \]

\[ = Q\left(\sqrt{2} \sqrt{\frac{y}{6}}\right) \]


Narum's Q function

\[ Q(a, b) = \int_0^\infty x \, e^{-\frac{x^2 + a^2}{2}} I_0(ax) \, dx \]
Receiver Operating Characteristic (ROC)

Using

\[ z = -\alpha^2 \ln(P_F) \]

we obtain

\[ P_D = Q\left(\sqrt{2\alpha^2}, \sqrt{-2 \ln(P_F)}\right) \]
Fluctuating Targets
Illustrating Targets

- Typical radar (e.g., aircrafts) have dimensions much larger than the wavelength.
  They should be considered as a large number of scattering points.

- The backscattered signals interfere in a random way.
  \( \Rightarrow \) The complex amplitude \( z \) of
  the signal at the output of the matched filter is a complex Gaussian random variable.

\( \Rightarrow \) The SNIR

\[
\gamma = \frac{|B|^2}{\sigma^2}
\]

is an exponential random variable with mean \( \bar{\sigma} \):

\[
\mathcal{P}_\gamma(x) = \begin{cases} \frac{x^{\frac{1}{2}}}{\bar{\sigma}} e^{-\frac{x}{2\bar{\sigma}}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}
\]
Decimation Variables

- Marginal probability density function

\[
p(y | H_1) = \left\{ \begin{array}{ll}
\int_{-\infty}^{\infty} p_{y|x}(y|x) \rho_0(x) \, dx \\
0 & \text{otherwise}
\end{array} \right.
\]

\[
p_{y|x}(y|x) = \begin{cases}
\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} I_0\left(\frac{2\sqrt{xy}}{\sigma}\right), & y > 0 \\
0, & \text{otherwise}
\end{cases}
\]

on obtains

\[
p(y | H_1) = \left\{ \begin{array}{ll}
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \int_{0}^{\infty} e^{-\alpha(x+\frac{y^2}{2\sigma^2})} I_0\left(\frac{2\sqrt{xy}}{\sigma}\right) \, d\alpha \\
0 & \text{otherwise}
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
\frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{y^2}{2\sigma^2}} \int_{0}^{\infty} e^{-\alpha(x+\frac{y^2}{2\sigma^2})} I_0\left(\frac{2\sqrt{xy}}{\sigma}\right) \, d\alpha, & y > 0 \\
0, & \text{otherwise}
\end{array} \right.
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0, & \text{otherwise}
\end{array} \right.
\]

\[
= \left\{ \begin{array}{ll}
0, & \text{otherwise}
\end{array} \right.
\]

\[
\int_{0}^{\infty} I_0(2\sqrt{ht}) e^{-\alpha t} \, dt = \frac{1}{2} e^{\frac{1}{2} \alpha^2}
\]

see Laplace transform of \( I_0(2\sqrt{ht}) \)
Probability of Detection

\[ P_D = \sum_{n=0}^{\infty} p(y | H_1) \, dy \]

\[ = \sum_{n=0}^{\infty} \frac{n}{\sigma^2 (n+\delta)} \, e^{-\frac{y}{\sigma^2 (n+\delta)}} \, dy \]

\[ = \left[ -e^{-\frac{y}{\sigma^2 (n+\delta)}} \right]_{n}^{\infty} \]

\[ = e^{-\frac{n}{\sigma^2 (n+\delta)}} \]
Receiver Operating Characteristic (ROC)

unit:

\[ n = -6^2 \ln (p_F) \]

one obtains

\[ p_D = e^{\frac{\sigma^2 \ln (p_F)}{\sigma^2 (1 + \delta)}} = p_F \frac{1}{1 + \delta} \]
Integration
Integration

Repeat the measurements $k$ times in order to get a more reliable detection result.

There are various methods to combine the results of the individual measurements.
**Coherent Integration**

- Coherent integration can be applied if the target is constant over the whole period of measurements.

- Optimum approach: apply a correlator to the whole sequence of measurements.

  \[ x = \sum_{n=1}^{K} x_n \]

  \[ \Rightarrow \text{SNIR improvement by a factor of } K \text{ (K times the energy)} \]

- In the case of a non-fluctuating target,

  \[ P_D = Q(\sqrt{2K\sigma^2}, \sqrt{-2\ln(P)}) \]
Incoherent Integration

- Incoherent integration can also be applied to time variant targets, in particular targets whose phase is randomly changing from measurement to measurement.

- Intuitive approach (suboptimum): summation of the incoherent correlator outputs $y_k$ of the individual measurements.

$$y = \sum_{k=1}^{K} y_k = \sum_{k=1}^{K} |x_k|^2$$
Probability of False Alarm

$y$ is a chi-squared random variable with $2k$ degrees of freedom in the case of a non-fluctuating target.

$$P_F = \Pr[y > \chi^2_k] = e^{-k/2} \sum_{n=0}^{k-1} \frac{1}{n!} \left( \frac{k}{2} \right)^n$$
Probability of Detection

$y$ is a noncentral chi-square random variable with $2K$ degrees of freedom in the case of a non-fluctuating target.

$$P_d = \Pr\left\{ y > y^* \right\} = Q_K \left( \sqrt{2K\bar{P}}, \sqrt{2\frac{K^2}{\sigma^2}} \right)$$

generalized Marcum's Q function
Detection results are combined.

A target is detected if there are at least $N$ hits in $K$ measurements.

Let $P_F$ and $P_D$ be the probabilities of false alarm and detection, respectively, for the individual measurements.
**Cumulative Probabilities**

- There are \( \binom{K}{N} = \frac{K!}{N! (K-N)!} \) possibilities to choose exactly \( N \) out of \( K \) measurements.

- Cumulative probability of false alarm
  
  \[
P_{cF} = \sum_{n=N}^{K} \binom{K}{n} P_f^n (1-P_f)^{K-n}
  \]

- Cumulative probability of detection
  
  \[
P_{cD} = \sum_{n=N}^{K} \binom{K}{n} P_d^n (1-P_d)^{K-n}
  \]
Special case: "N=1 out of K"

- Cumulative probability of detection
  
  \[ P_{CD} = 1 - (1 - P_D)^K \]

  probability of no hit

- Cumulative probability of false alarm
  
  \[ P_{CF} = 1 - (1 - P_F)^K \]

- Linear approximation for small \( P_F \)
  
  \[ P_{CF} \approx K P_F \]

  In other words: for small \( P_F \), "at least 1 out of K" is approximately equal to "exactly 1 out of K"

  \[
  P_{CF} \approx \left( \frac{K}{1} \right) P_F \left( 1 - P_F \right)^{K-1} \approx 1 \\
  \approx K P_F
  \]
$K=4 \quad \xi=7$