Matrices

Four Subspaces

- The column space is spanned by the columns of $\underline{A}$.
- The nullspace contains all vectors $\underline{x}$ with $\underline{A} \cdot \underline{x} = 0$.
- The row space is spanned by the columns of $\underline{A}^{*T}$.
- The left nullspace contains all vectors $\underline{y}$ with $\underline{A}^{*T} \cdot \underline{y} = 0$.

Eigenvalues and Eigenvectors

- eigenvalue problem

$$\underline{A} \cdot \underline{x} = \underline{\lambda}\ \underline{x}$$

$$\Leftrightarrow (\underline{A} - \underline{\lambda}\ \underline{E}) \cdot \underline{x} = 0$$

$\underline{x}$ → eigenvector

$\underline{E}$ → identity matrix

$\underline{\lambda}$ → eigenvalue

$\underline{A}$ → square matrix

- eigenvalues are the roots of the characteristic polynomial

$$\det(\underline{A} - \underline{\lambda}\ \underline{E}) = 0$$

- in the following the eigenvectors will be normalized

$$\underline{u} = \frac{\underline{x}}{\|\underline{x}\|}$$

Determinant of a Matrix

- factorization of the characteristic polynomial

$$\det(\underline{A} - \underline{\lambda}\, E) = \prod_n (\underline{\lambda}_n - \underline{\lambda})$$

↳ eigenvalues

set $\underline{\lambda} = 0$

$$\det(\underline{A}) = \prod_n \underline{\lambda}_n$$

product of the eigenvalues

- product of matrices:

$$\det(\underline{A} \cdot \underline{B}) = \det(\underline{A}) \cdot \det(\underline{B})$$

Trace of a Matrix

- definition: sum of the diagonal elements of a square matrix $\underline{A}$

$$\text{trace}(\underline{A}) = \sum_n [\underline{A}]_{n,n}$$

$$\Rightarrow \text{trace}\Big(\underbrace{\overset{N\times M}{\underline{B}} \cdot \overset{M\times N}{\underline{C}'}}_{N\times N}\Big)$$

$$= \sum_n [\underline{B}\cdot\underline{C}']_{n,n}$$

$$= \sum_n \sum_m [\underline{B}]_{n,m} \, [\underline{C}']_{m,n}$$

$$= \sum_m \sum_n [\underline{C}']_{m,n} \, [\underline{B}]_{n,m}$$

$$= \sum_m [\underline{C}'\cdot\underline{B}]_{m,m}$$

$$= \text{trace}\Big(\underbrace{\underline{C}'\cdot\underline{B}}_{M\times M}\Big)$$

- factorization of the characteristic polynomial

$$\det(\underline{A} - \underline{\lambda} E) = \prod_{n=1}^{N} (\underline{\lambda}_n - \underline{\lambda})$$

coefficient of $\underline{\lambda}^{N-1}$

- on the left hand side

$$\sum_n (-1)^{N-1} [\underline{A}]_{n,n} = (-1)^{N-1} \operatorname{trace}(\underline{A})$$

- on the right hand side

$$\sum_n (-1)^{N-1} \underline{\lambda}_n = (-1)^{N-1} \sum_n \underline{\lambda}_n$$

$\Rightarrow \operatorname{trace}(\underline{A}) = \sum_n \underline{\lambda}_n$ sum of the eigenvalues

Similar Matrices

- definition : two square matrices $\underline{A}$ and $\underline{B}$ are similar iff there exists an invertible matrix $\underline{U}$ such that

$$\underline{U}^{-1} \cdot \underline{A} \cdot \underline{U} = \underline{B}$$

$$\underline{A} \cdot \underline{U} = \underline{U} \cdot \underline{B}$$

$$\underline{A} = \underline{U} \cdot \underline{B} \cdot \underline{U}^{-1}$$

- let $\underline{y}$ be an eigenvector of $\underline{B}$ with corresponding eigenvalue λ

$$\Rightarrow \quad \underline{A} \cdot \underbrace{\underline{U} \cdot \underline{y}}_{\underline{x}} = \underline{U} \cdot \underline{B} \cdot \underline{y}$$

$$= \lambda \, \underbrace{\underline{U} \cdot \underline{y}}_{\underline{x}}$$

$\Rightarrow$ $\underline{x} = \underline{U} \cdot \underline{y}$ is an eigenvector of $\underline{A}$ with corresponding eigenvalue λ

- Similar matrices have the same eigenvalues.

- Similar matrices have the same rank.

Unitary Matrices

- definition: a square matrix $\underline{U}$ is unitary iff its columns (rows) are orthonormal

$$\underline{U}^{*T} \cdot \underline{U} = E \qquad (\underline{U} \cdot \underline{U}^{*T} = E)$$

$$\Leftrightarrow \quad \underline{U}^{-1} = \underline{U}^{*T}$$

- a product of two unitary matrices $\underline{U}_1$ and $\underline{U}_2$ is again a unitary matrix:

$$\begin{aligned}(\underline{U}_1 \cdot \underline{U}_2)^{*T} \cdot (\underline{U}_1 \cdot \underline{U}_2) &= \underline{U}_2^{*T} \cdot \underline{U}_1^{*T} \cdot \underline{U}_1 \cdot \underline{U}_2 \\ &= \underline{U}_2^{*T} \cdot \underline{U}_2 \\ &= E\end{aligned}$$

- multiplication with a unitary matrix $\underline{U}$ does not change the norm of a vector $\underline{x}$:

$$\begin{aligned}\|\underline{U} \cdot \underline{x}\|^2 &= (\underline{U} \cdot \underline{x})^{*T} \cdot (\underline{U} \cdot \underline{x}) \\ &= \underline{x}^{*T} \cdot \underline{U}^{*T} \cdot \underline{U} \cdot \underline{x} \\ &= \underline{x}^{*T} \cdot \underline{x} \\ &= \|\underline{x}\|^2\end{aligned}$$

• determinant of a unitary matrix:

$$
\begin{aligned}
|\det(\underline{U})|^2 &= (\det(\underline{U}))^* \cdot \det(\underline{U}) \\
&= \det(\underline{U}^*) \cdot \det(\underline{U}) \\
&= \det(\underline{U}^{*T}) \cdot \det(\underline{U}) \\
&= \det(\underline{U}^{*T} \cdot \underline{U}) \\
&= \det(E) \\
&= 1
\end{aligned}
$$

$$
\Rightarrow |\det(\underline{U})| = 1
$$

Kronecker Product

- definition:

$$\begin{pmatrix} \underline{a}_{1,1} & \cdots & a_{1,N} \\ \vdots & & \vdots \\ \underline{a}_{M,1} & \cdots & \underline{a}_{M,N} \end{pmatrix} \otimes \underline{B} = \begin{pmatrix} \underline{a}_{1,1}\,\underline{B} & \cdots & \underline{a}_{1,N}\,\underline{B} \\ \vdots & & \vdots \\ \underline{a}_{M,1}\,\underline{B} & \cdots & \underline{a}_{M,N}\,\underline{B} \end{pmatrix}$$

- rules:

  - $\underline{c}\,(\underline{A} \otimes \underline{B}) = (\underline{c}\,\underline{A}) \otimes \underline{B} = \underline{A} \otimes (\underline{c}\,\underline{B})$
  - $\underline{A} \otimes (\underline{B} \otimes \underline{C}') = (\underline{A} \otimes \underline{B}) \otimes \underline{C}'$
  - $(\underline{A} \otimes \underline{B})^{*T} = \underline{A}^{*T} \otimes \underline{B}^{*T}$
  - $(\underline{A} \otimes \underline{B}) \cdot (\underline{C}' \otimes \underline{D}) = (\underline{A} \cdot \underline{C}') \otimes (\underline{B} \cdot \underline{D})$
  - $(\underline{A} \otimes \underline{B})^{-1} = \underline{A}^{-1} \otimes \underline{B}^{-1}$
  - $(\underline{A} + \underline{B}) \otimes \underline{C}' = \underline{A} \otimes \underline{C}' + \underline{B} \otimes \underline{C}$
  - $\underline{A} \otimes (\underline{B} + \underline{C}) = \underline{A} \otimes \underline{B} + \underline{A} \otimes \underline{C}$
  - $\mathrm{vec}(\underline{A} \cdot \underline{B} \cdot \underline{C}') = (\underline{C}^T \otimes \underline{A}) \cdot \mathrm{vec}(\underline{B})$

Woodbury Matrix Identity

$$(A+B\cdot C\cdot D)^{-1} = A^{-1} - A^{-1}\cdot B\cdot(C^{-1}+D\cdot A^{-1}\cdot B)^{-1}\cdot D\cdot A^{-1}$$