Hermitian Matrices

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definition: a matrix $\underline{A}$ is Hermitian iff

$$\underline{A}^{*T} = \underline{A}$$

Real Eigenvalues

Let $\underline{u}$ be a normalised eigenvector of the Hermitian matrix $\underline{A}$ with corresponding eigenvalue λ.

$$\lambda = \underline{u}^{*T} \cdot \lambda \cdot \underline{u} = \underline{u}^{*T} \cdot \underline{A} \cdot \underline{u}$$

$$\Rightarrow \lambda^{*T} = \left(\underline{u}^{*T} \cdot \underline{A} \cdot \underline{u} \right)^{*T}$$

$$= \underline{u}^{*T} \cdot \underline{A}^{*T} \cdot \underline{u}$$

$$= \underline{u}^{*T} \cdot \underline{A} \cdot \underline{u}$$

$$= \underline{u}^{*T} \cdot \lambda \cdot \underline{u}$$

$$= \lambda$$

The eigenvalues of Hermitian matrices are real.

Orthogonal Eigenvectors

Let $\underline{u}_1$ and $\underline{u}_2$ be two (normalized) eigenvectors of the Hermitian matrix $\underline{A}$ with corresponding distinct real eigenvalues λ_1 and λ_2.

$$
\begin{aligned}
\lambda_2 \cdot \underline{u}_1^{*T} \cdot \underline{u}_2 &= \underline{u}_1^{*T} \cdot \lambda_2 \cdot \underline{u}_2 \\
&= \underline{u}_1^{*T} \cdot \underline{A} \cdot \underline{u}_2 \\
&= \left(\left(\underline{u}_1^{*T} \cdot \underline{A} \cdot \underline{u}_2 \right)^{*T} \right)^{*T} \\
&= \left(\underline{u}_2^{*T} \cdot \underline{A}^{*T} \cdot \underline{u}_1 \right)^{*T} \\
&= \left(\underline{u}_2^{*T} \cdot \underline{A} \cdot \underline{u}_1 \right)^{*T} \\
&= \left(\underline{u}_2^{*T} \cdot \lambda_1 \cdot \underline{u}_1 \right)^{*T} \\
&= \underline{u}_1^{*T} \cdot \lambda_1 \cdot \underline{u}_2 \\
&= \lambda_1 \cdot \underline{u}_1^{*T} \cdot \underline{u}_2
\end{aligned}
$$

$$\Rightarrow \underline{u}_1^{*T} \cdot \underline{u}_2 = 0 \quad \text{as} \quad \lambda_1 \neq \lambda_2$$

The (normalized) eigenvectors of Hermitian matrices corresponding to distinct eigenvalues are orthogonal (orthonormal).

Spectral Theorem

Every Hermitian matrix $\underline{A}$ can be diagonalized by a unitary matrix $\underline{U}$, i.e.,

$$\underline{U}^{*T} \cdot \underline{A} \cdot \underline{U} = \Lambda ,$$

where Λ is a real diagonal matrix.

If the Hermitian matrix $\underline{A}$ has distinct eigenvalues $\lambda_n, n = 1 \ldots N$, with corresponding normalized eigenvectors $\underline{u}_n, n = 1 \ldots N$, the spectral theorem is immediate.
Choose

$$\underline{U} = (\underline{u}_1, \ldots, \underline{u}_N)$$

and

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} .$$

But the spectral theorem also holds if the Hermitian matrix $\underline{A}$ has repeated eigenvalues.

Eigendecomposition

$$\underline{U}^{*T} \cdot \underline{A} \cdot \underline{U} = \Lambda$$

$$\underline{A} \cdot \underline{U} = \underline{U} \cdot \Lambda$$

$$\underline{A} = \underline{U} \cdot \Lambda \cdot \underline{U}^{*T}$$

- The Hermitian matrix $\underline{A}$ is similar to the diagonal matrix Λ.
- The rank R of a Hermitian matrix $\underline{A}$ is equal to the number of nonzero eigenvalues (considering their multiplicities), i.e., the number of nonzero diagonal elements of Λ.

Four Subspaces

- eigendecomposition of the Hermitian matrix $\underline{A}$ with rank R:

$$\underline{A} = \left(\underline{u}_1, \dots, \underline{u}_R, \underline{u}_{R+1}, \dots, \underline{u}_N\right) \cdot \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_R & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{u}_1^{*T} \\ \vdots \\ \underline{u}_R^{*T} \\ \underline{u}_{R+1}^{*T} \\ \vdots \\ \underline{u}_N^{*T} \end{pmatrix}$$

- The column space is identical to the row space.
  $\underline{u}_1, \dots, \underline{u}_R$ is an orthonormal basis.
  The dimension is R.
- The nullspace is identical to the left nullspace.
  $\underline{u}_{R+1}, \dots, \underline{u}_N$ is an orthonormal basis.
  The dimension is $N-R$.

Inverse

The inverse of a full rank $R=N$ Hermitian matrix

$$\underline{A} = \underline{U} \cdot \Lambda \cdot \underline{U}^{*T}$$

is

$$\underline{A}^{-1} = \underline{U} \cdot \Lambda^{-1} \cdot \underline{U}^{*T} .$$

proof:

$$\underline{A}^{-1} \cdot \underline{A} = \underline{U} \cdot \Lambda^{-1} \cdot \underline{U}^{*T} \cdot \underline{U} \cdot \Lambda \cdot \underline{U}^{*T}$$

$$= \underline{U} \cdot \Lambda^{-1} \cdot \Lambda \cdot \underline{U}^{*T}$$

$$= \underline{U} \cdot \underline{U}^{*T}$$

$$= \underline{E}$$