Normal Distribution

Multivariate Complex Normal Distribution

- probability density function:

$$p(\underline{x}) = \frac{1}{\pi^N \det(\underline{C}_{xx})} e^{-(\underline{x}-\underline{\mu}_x)^{*T} \cdot \underline{C}_{xx}^{-1} \cdot (\underline{x}-\underline{\mu}_x)}$$

- $\underline{x} = \begin{pmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_N \end{pmatrix}$: N dimensional random vector

- $\underline{C}_{xx}$: $N \times N$ Hermitian positive definite matrix

  eigendecomposition:

$$\underline{C}_{xx} = \underline{U} \cdot \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} \cdot \underline{U}^{*T}$$

- $\underline{\mu}_x$: N dimensional vector

· normalization

$$\int p(\underline{x})\,d\underline{x} = \int \frac{1}{\pi^N \det(\underline{C}_{xx})}\, e^{-(\underline{x}-\underline{\mu}_x)^{*T}\cdot \underline{C}_{xx}^{-1}\cdot(\underline{x}-\underline{\mu}_x)}\, d\underline{x}$$

| substitution $\underline{x} = \underline{U}\cdot\underline{y} + \underline{\mu}_x$

$$= \int \frac{1}{\pi^N \det(\underline{C}_{xx})}\, e^{-\underline{y}^{*T}\cdot \underline{U}^{*T}\cdot \underline{C}_{xx}^{-1}\cdot \underline{U}\cdot \underline{y}}\, d\underline{y}$$

$$= \int \frac{1}{\pi^N \det(\Lambda)}\, e^{-\underline{y}^{*T}\cdot \Lambda^{-1}\cdot \underline{y}}\, d\underline{y}$$

$$= \prod_n \frac{1}{\pi \lambda_n} \underbrace{\int e^{-\frac{|y_n|^2}{\lambda_n}}\, dy_n}_{\pi \lambda_n}$$

$$= 1$$

- mean:

$$E\{\underline{x}\} = \int \underline{x}\; p(\underline{x})\, d\underline{x}$$

$$= \int \frac{\underline{x}}{\pi^N \det(\underline{C}_{xx})}\, e^{-(\underline{x}-\underline{\mu}_x)^{*T}\cdot \underline{C}_{xx}^{-1}\cdot(\underline{x}-\underline{\mu}_x)}\, d\underline{x}$$

| substitution $\underline{x} = \underline{U}\cdot \underline{y} + \underline{\mu}_x$

$$= \int \frac{\underline{U}\cdot\underline{y} + \underline{\mu}_x}{\pi^N \det(\underline{C}_{xx})}\, e^{-\underline{y}^{*T}\cdot \underline{U}^{*T}\cdot \underline{C}_{xx}^{-1}\cdot \underline{U}\cdot \underline{y}}\, d\underline{y}$$

$$= \int \frac{\underline{U}\cdot\underline{y} + \underline{\mu}_x}{\pi^N \det(\Lambda)}\, e^{-\underline{y}^{*T}\cdot \Lambda^{-1}\cdot \underline{y}}\, d\underline{y}$$

| symmetry

$$= \int \frac{\underline{\mu}_x}{\pi^N \det(\Lambda)}\, e^{-\underline{y}^{*T}\cdot \Lambda^{-1}\cdot \underline{y}}\, d\underline{y}$$

$$= \underline{\mu}_x \prod_n \frac{1}{\pi \lambda_n} \underbrace{\int e^{-\frac{|y_n|^2}{\lambda_n}}\, dy_n}_{\pi \lambda_n}$$

$$= \underline{\mu}_x$$

• covariance matrix:

$$E\{(\underline{x}-\underline{\mu}_x)\cdot(\underline{x}-\underline{\mu}_x)^{*T}\} = \int (\underline{x}-\underline{\mu}_x)\cdot(\underline{x}-\underline{\mu}_x)^{*T}\, p(\underline{x})\, d\underline{x}$$

$$= \int \frac{(\underline{x}-\underline{\mu}_x)\cdot(\underline{x}-\underline{\mu}_x)^{*T}}{\pi^N \det(\underline{C}_{xx})}\, e^{-(\underline{x}-\underline{\mu}_x)^{*T}\cdot \underline{C}_{xx}^{-1}\cdot(\underline{x}-\underline{\mu}_x)}\, d\underline{x}$$

| substitution $\underline{x} = \underline{U}\cdot\underline{y} + \underline{\mu}_x$

$$= \int \frac{\underline{U}\cdot\underline{y}\cdot\underline{y}^{*T}\cdot\underline{U}^{*T}}{\pi^N \det(\underline{C}_{xx})}\, e^{-\underline{y}^{*T}\cdot\underline{U}^{*T}\cdot\underline{C}_{xx}^{-1}\cdot\underline{U}\cdot\underline{y}}\, d\underline{y}$$

$$= \underline{U}\cdot \int \frac{\underline{y}\cdot\underline{y}^{*T}}{\det(\Lambda)}\, e^{-\underline{y}^{*T}\cdot\Lambda^{-1}\cdot\underline{y}}\, d\underline{y}\cdot\underline{U}^{*T}$$

| symmetry

$$= \underline{U}\cdot\begin{pmatrix} \int \frac{|y_1|^2}{\pi\lambda_1} e^{-\frac{|y_1|^2}{\lambda_1}}\, dy_1 & & 0 \\ & \ddots & \\ 0 & & \int \frac{|y_N|^2}{\pi\lambda_N} e^{-\frac{|y_N|^2}{\lambda_N}}\, dy_N \end{pmatrix}\cdot\underline{U}^{*T}$$

$$= \underline{U}\cdot\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}\cdot\underline{U}^{*T}$$

$$= \underline{C}_{xx}$$

White Gaussian Noise

- zero mean: $\underline{\mu}_n = 0$

- the covariance matrix is equal to the correlation matrix:

$$\underline{C}_{nn} = \underline{R}_{nn} = \sigma^2 E$$

- probability density function:

$$p(\underline{n}) = \frac{1}{\pi^N \sigma^{2N}} e^{-\frac{\underline{n}^{*T} \cdot \underline{n}}{\sigma^2}}$$

$$= \frac{1}{\pi^N \sigma^{2N}} e^{-\frac{\|\underline{n}\|^2}{\sigma^2}}$$