Random Signals

Random Signals

In the following all signals (not only the noise) will be considered random.

assumptions:

- zero mean

$$E\{\underline{s}_{RP,k}\} = 0$$

$$E\{\underline{n}_n\} = 0$$

$\Rightarrow$ correlation and covariance are the same

- uncorrelated sources

$$E\{\underline{s}_{RP,k}\ \underline{s}^*_{RP,\ell}\} = \begin{cases} \sigma^2_{RP,k} & k = \ell \quad \text{source power} \\ 0 & k \neq \ell \end{cases}$$

- white noise

$$E\{\underline{n}_m\ \underline{n}^*_n\} = \begin{cases} \sigma^2 & m = n \quad \text{noise power} \\ 0 & m \neq n \end{cases}$$

Correlations

correlations of the received signals

$$\underline{r}_{m,n} = E\{\underline{e}_m \underline{e}_n^*\}$$

$$= E\Big\{\Big(\underbrace{\sum_k \underline{a}_{m,k}\,\underline{s}_{RP,k} + \underline{n}_m}_{\underline{r}_m}\Big)\Big(\underbrace{\sum_\ell \underline{a}_{n,\ell}^*\,\underline{s}_{RP,\ell}^* + \underline{n}_n^*}_{\underline{r}_n^*}\Big)\Big\}$$

$$= \sum_k \underline{a}_{m,k}\,\underline{a}_{n,k}^*\,E\{\underline{s}_{RP,k}\,\underline{s}_{RP,k}^*\}$$

$$+ \sum_{k\neq\ell} \underline{a}_{m,k}\,\underline{a}_{n,\ell}^*\,\underbrace{E\{\underline{s}_{RP,k}\,\underline{s}_{RP,\ell}^*\}}_{0}$$

$$+ \sum_k \underline{a}_{m,k}\,\underbrace{E\{\underline{s}_{RP,k}\,\underline{n}_n^*\}}_{0}$$

$$+ \sum_\ell \underline{a}_{n,\ell}^*\,\underbrace{E\{\underline{n}_m\,\underline{s}_{RP,\ell}^*\}}_{0}$$

$$+ E\{\underline{n}_m\,\underline{n}_n^*\}$$

$$= \begin{cases} \sum_k |\underline{a}_{n,k}|^2\,\sigma_{RP,k}^2 + \sigma^2 & m = n \\ \sum_k \underline{a}_{m,k}\,\underline{a}_{n,k}^*\,\sigma_{RP,k}^2 & m \neq n \end{cases}$$

$$= \begin{cases} \sum_k \sigma_{RP,k}^2 + \sigma^2 & m = n \\ \sum_k e^{j\langle \vec{r}_m - \vec{r}_n, \vec{\beta}_k\rangle}\,\sigma_{RP,k}^2 & m \neq n \end{cases}$$

Co-Array

- The correlations only depend on the lags $\vec{r}_m - \vec{r}_n$, not on the absolute positions $\vec{r}_m$ and $\vec{r}_n$ of the antenna elements.

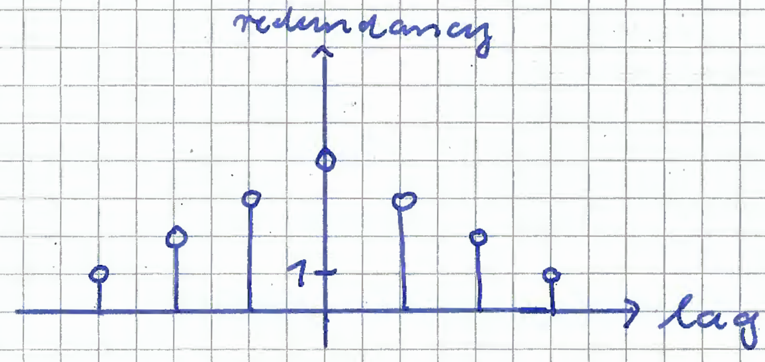
- redundancy
  Different pairs of antenna elements may yield the same lag and thus the same correlation.

- co-array
  characterized by the lags and their redundancies

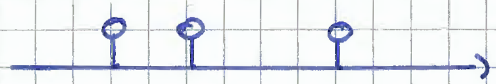
Example

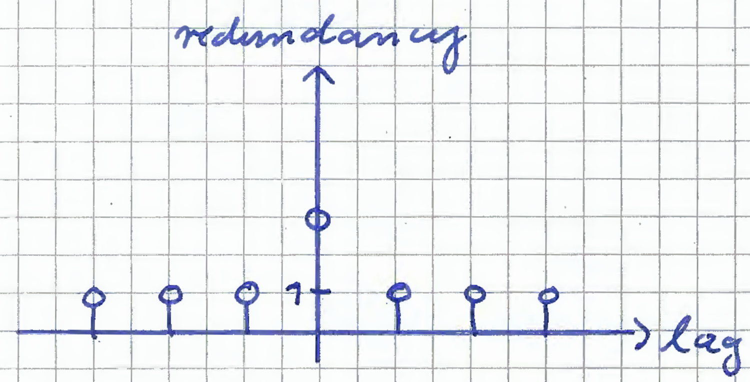
array | co-array: autocorrelation function of the array

uniform linear array, $N = 4$



sparse array, $N = 3$





$\Rightarrow$ the sparse array yields the same information but requires less antenna elements

Sparse (Linear) Arrays

antenna elements placed
on a regular grid
=> thinned uniform
(linear) arrays

discussion based on the co-array

- minimum redundancy arrays:
  gapless co-array with
  minimum number of
  antenna elements

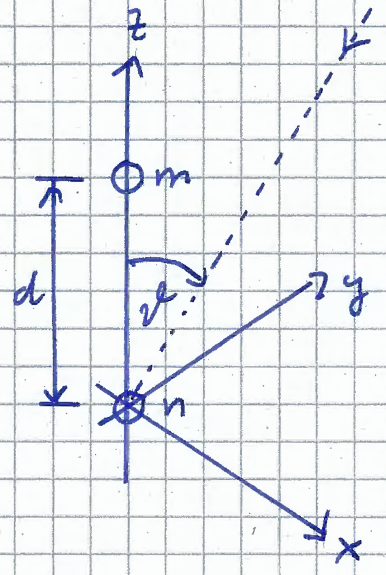
- non-redundant arrays:
  no lag (except the zero lag)
  shows up several times

- perfect arrays:
  non-redundant arrays
  with gapless co-array
  (exist only in rare cases)

the above example was
a perfect array

Omnidirectional Noise

- two antenna elements:



- steering factor (antenna element n at the reference point):

$$\underline{a} = e^{j d \beta \cos(\vartheta)}$$

- received noise:

$$\underline{n}_m = \underline{a} \cdot \underline{n}_n$$

- uniform probability density function of the direction of arrival:

$$p(\vartheta, \varphi) = \frac{\sin(\vartheta)}{4\pi}$$

• noise correlation:

$$E\{\underline{n}_m \underline{n}_n^*\} = E\{\underline{a}_m \underline{n}_n \underline{n}_n^*\} = \sigma^2 E\{\underline{a}_m\}$$

$$= \sigma^2 \int_0^{\pi} \int_0^{2\pi} \frac{\sin(\vartheta)}{4\pi} e^{j d \beta \cos(\vartheta)} \, d\varphi \, d\vartheta$$

$$= \frac{\sigma^2}{2} \int_0^{\pi} \sin(\vartheta) e^{j d \beta \cos(\vartheta)} \, d\vartheta$$

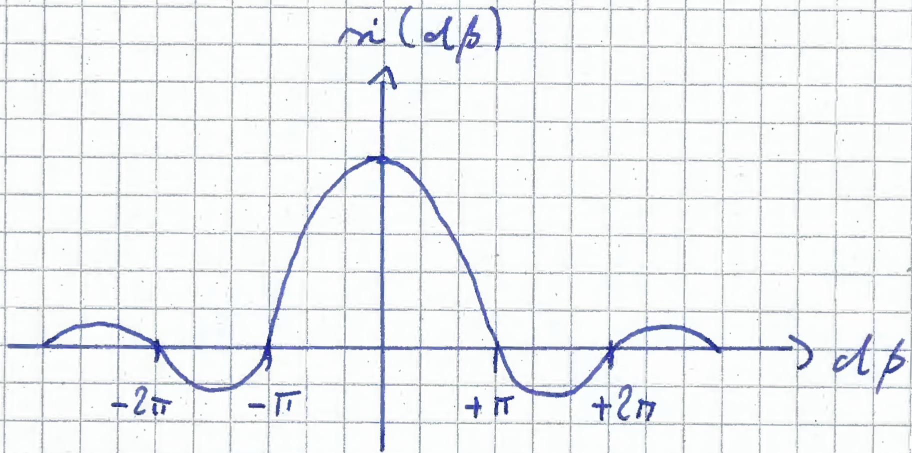
| substitution $x = \cos(\vartheta)$

$$= \frac{\sigma^2}{2} \int_{-1}^{+1} e^{j d \beta x} \, dx$$

$$= \frac{\sigma^2}{2} \int_{-1}^{+1} \cos(d \beta x) \, dx$$

$$= \frac{\sigma^2}{2} \left[\frac{\sin(d \beta x)}{d \beta} \right]_{-1}^{+1}$$

$$= \frac{\sigma^2}{2} \underbrace{\frac{\sin(d \beta)}{d \beta}}_{\mathrm{si}(d\beta)}$$



⇒ no correlation if d is an integer multiple of $\lambda/2$