Antenna Arrays

Antenna Arrays

- antenna arrays consist of several antenna elements

- in the following
  - identical and
  - identically oriented

  antenna elements

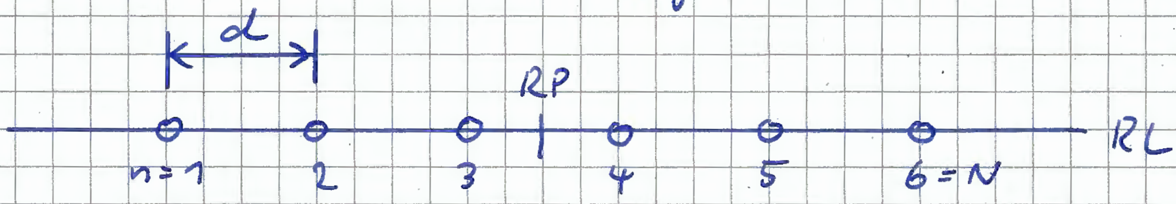
- mutual couplings will be neglected

- for simplicity omnidirectional antenna elements assumed

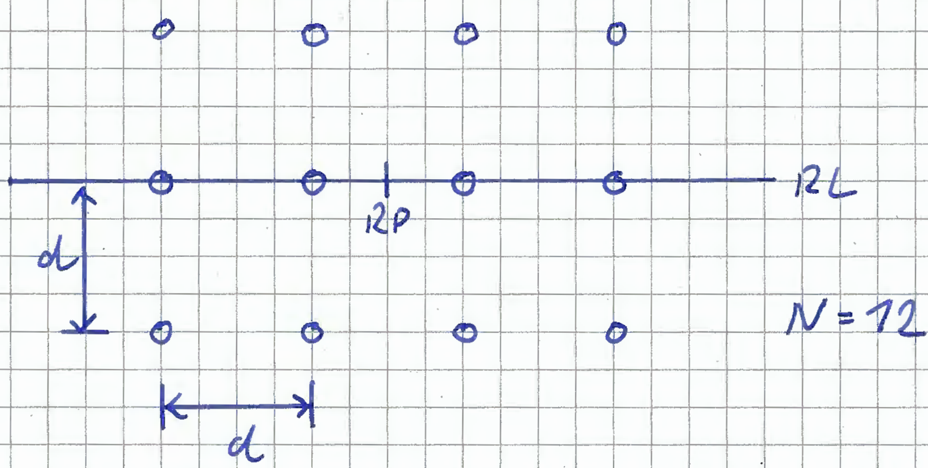
$\Rightarrow$ sampling of the wavefield in the spatial domain

Configurations

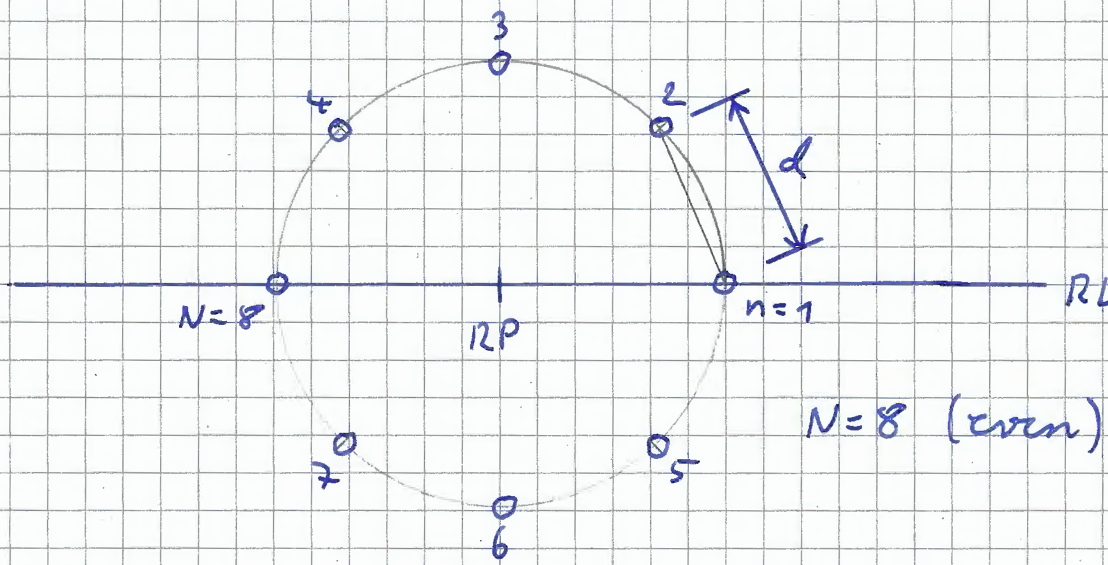
- Uniform Linear Array (ULA)



- Uniform Planar Array (UPA)



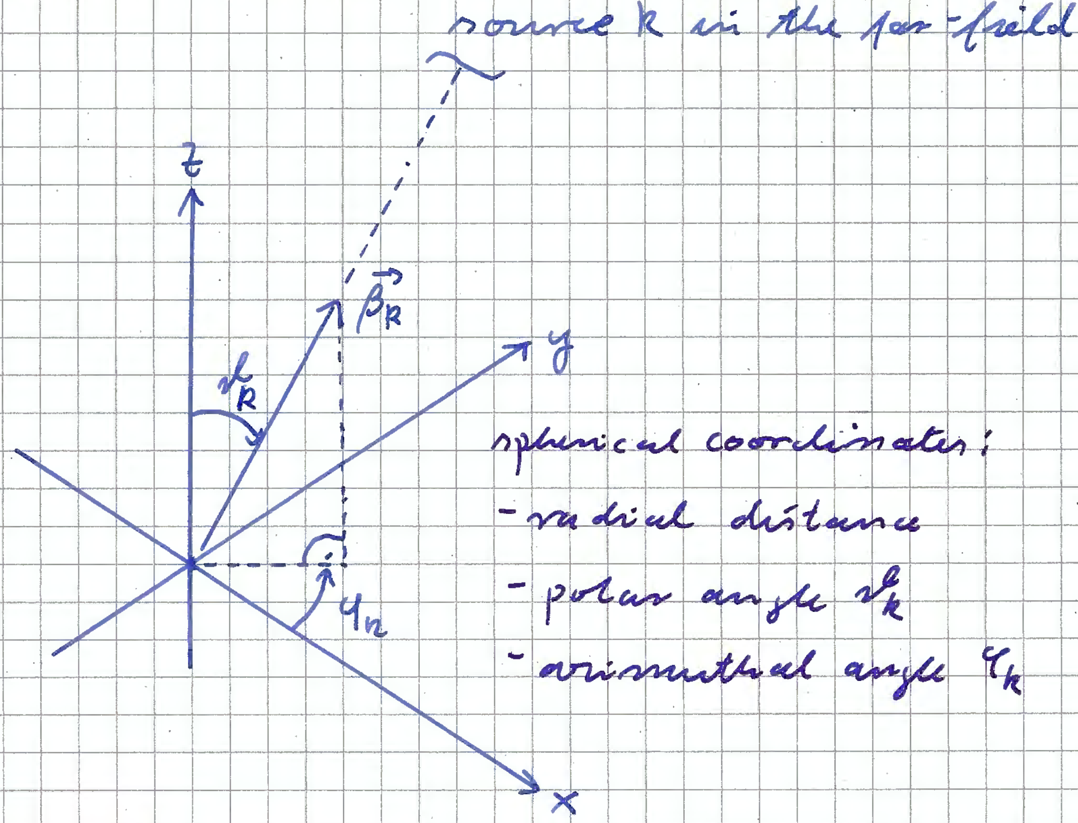
- Uniform Circular Array (UCA)



RP : Reference Point

RL : Reference Line
(for measuring directions of arrival)

Wavenumber Vector



spherical coordinates:
- radial distance
- polar angle ϑ_k
- azimuthal angle φ_k

- wavenumber vector:

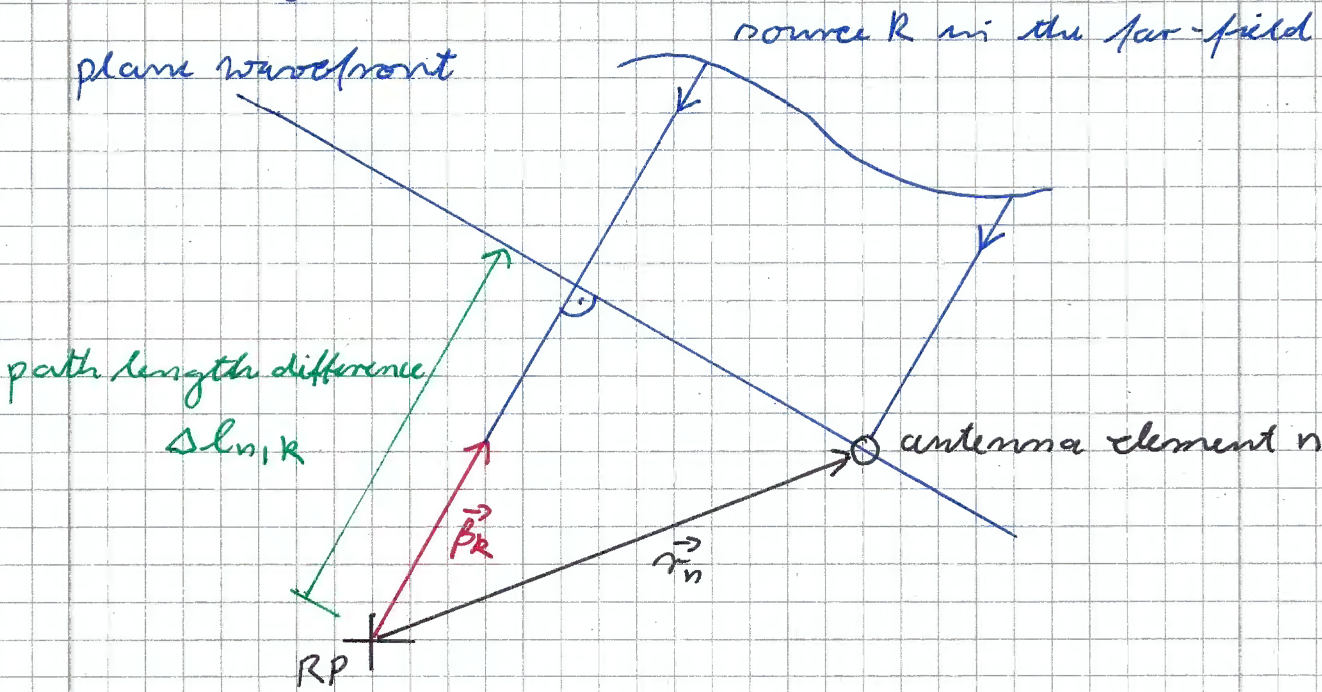
$$\vec{\beta}_k = \begin{pmatrix} \beta_{x,k} \\ \beta_{y,k} \\ \beta_{z,k} \end{pmatrix} = \beta \underbrace{\begin{pmatrix} \sin(\vartheta_k)\cos(\varphi_k) \\ \sin(\vartheta_k)\sin(\varphi_k) \\ \cos(\vartheta_k) \end{pmatrix}}_{\text{direction cosines}}$$

  - points towards the source k
  - norm $\|\vec{\beta}_k\| = \beta = \frac{2\pi}{\lambda}$

- direction of arrival characterized by
  - the solid angle (ϑ_k, φ_k),
  - the (components) of the wavenumber vector $\vec{\beta}_k$ or
  - the direction cosines

- examples in the following for 2D (only azimuthal angle φ_k), can be easily extended to 3D

Steering Factor



- phase shift:

$$2\pi \frac{\Delta l_{n,k}}{\lambda} = \langle \vec{r}_n, \vec{\beta}_k \rangle$$

- steering factor:

$$\underline{a}_{n,k} = e^{j\langle \vec{r}_n, \vec{\beta}_k \rangle}$$

- array manifold vector, N antenna elements:

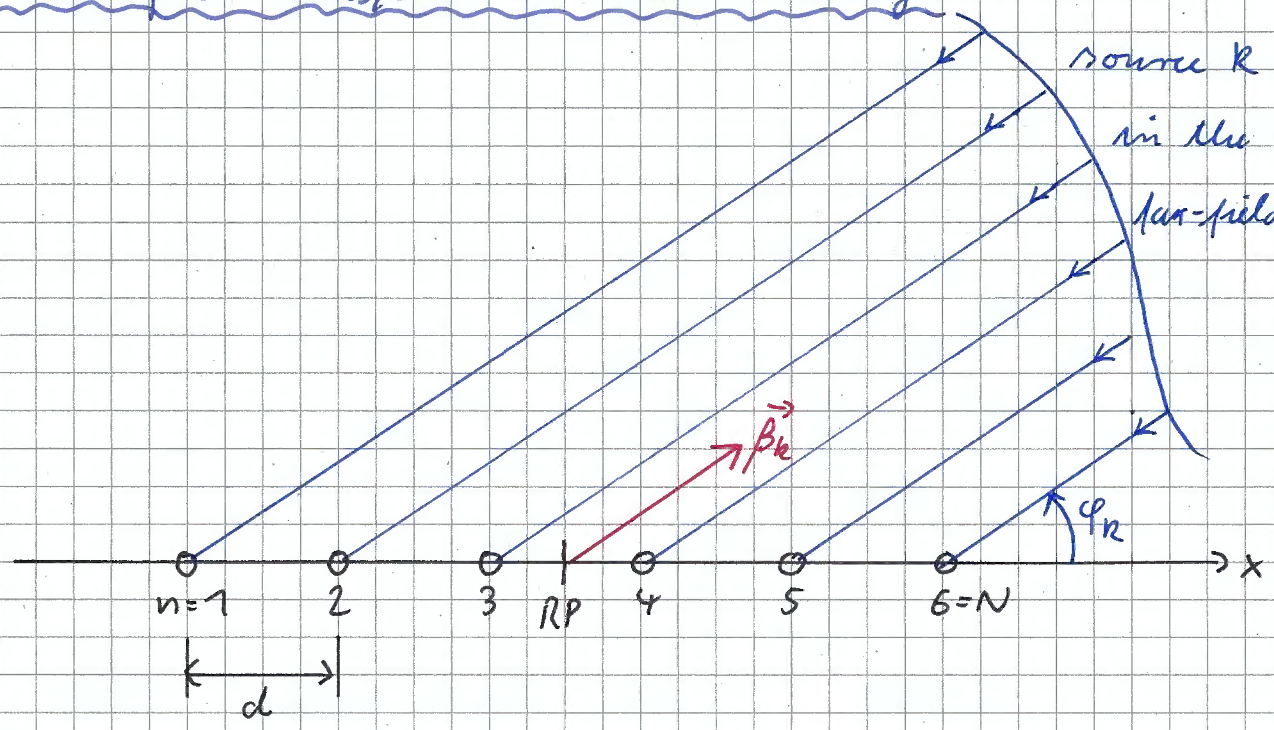
$$\underline{a}_k = \begin{pmatrix} \underline{a}_{1,k} \\ \vdots \\ \underline{a}_{N,k} \end{pmatrix}$$

- array manifold matrix, K sources:

$$\underline{A} = (\underline{a}_1, \ldots, \underline{a}_K) = \begin{pmatrix} \underline{a}_{1,1} & \cdots & \underline{a}_{1,K} \\ \vdots & & \vdots \\ \underline{a}_{N,1} & \cdots & \underline{a}_{N,K} \end{pmatrix}$$

$N \times K$ matrix

Example: Uniform Linear Array



- position of the n-th antenna element:

$$\vec{r}_n = \begin{pmatrix} n - \frac{N+1}{2} \\ 0 \\ 0 \end{pmatrix} d$$

- steering factor:

$$a_{n,k} = e^{j\left(n-\frac{N+1}{2}\right) d \, \beta \cos(\varphi_k)}$$

$$= e^{j\left(n-\frac{N+1}{2}\right) d \, \beta_{x,k}} = \underline{\varepsilon}^{\left(n-\frac{N+1}{2}\right)}$$

using $\underline{\varepsilon} = e^{j d \beta_{x,k}} = e^{j d \beta \cos(\varphi_k)}$

- direction cosine: $\cos(\varphi_k)$

- $\beta_{x,k}$ is bandlimited:

$$-\beta \leq \beta_{x,k} \leq +\beta$$

$\Rightarrow$ $\beta_{x,k}$ represents a propagating wavefront

• array manifold vector:

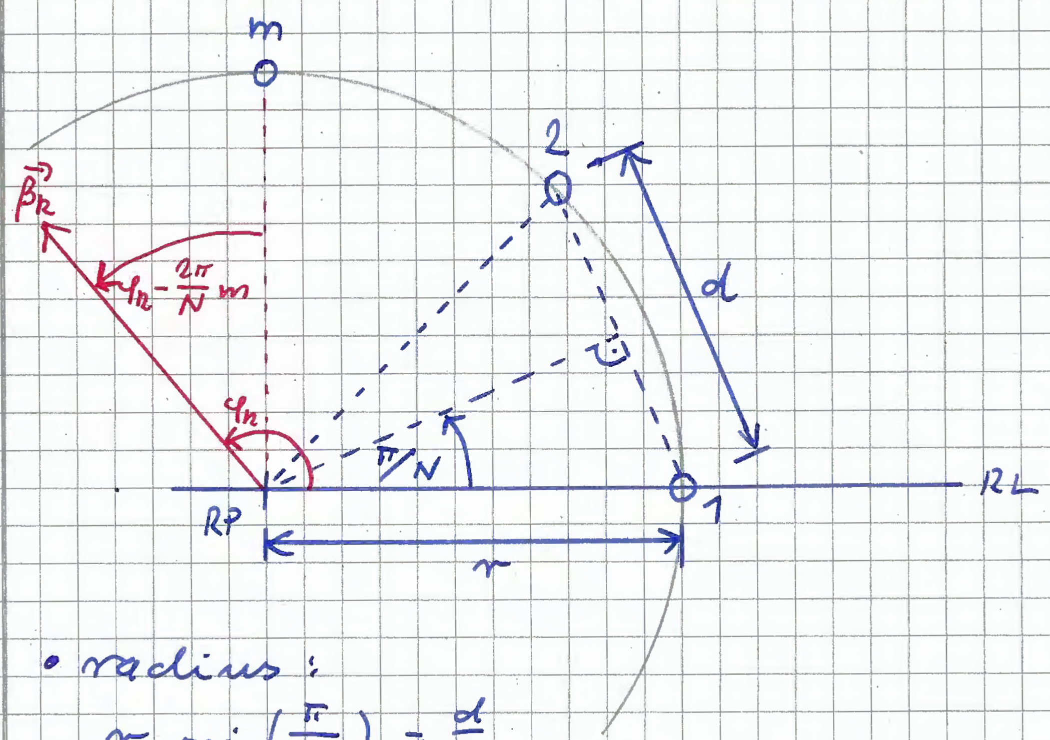
$$\underline{a}_k = \begin{pmatrix} \underline{\varepsilon}_k^{-\frac{N-1}{2}} \\ \underline{\varepsilon}_k^{-\frac{N-3}{2}} \\ \vdots \\ \underline{\varepsilon}_k^{+\frac{N-3}{2}} \\ \underline{\varepsilon}_k^{+\frac{N-1}{2}} \end{pmatrix}$$

• array manifold vectors $\underline{a}_k$ of centrosymmetric arrays like the uniform linear array are conjugate centrosymmetric:

$$\underbrace{\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}}_{\text{exchange matrix } \Pi} \cdot \underline{a}_k^* = \underline{a}_k$$

```
function A = ula(phi, N, d)
%ULA compute the array manifold matrix of a uniform linear array
%   A: N x K array manifold matrix
%   phi: vector with K directions of arrival
%   N: number of antenna elements
%   d: distance of antenna elements normalized to the wavelength

K = numel(phi); % number of sources

% the steering factors are exp(j*2*pi*d*(n-(N+1)/2)*cos(phi))
A = exp(j*2*pi*d*repmat((1:N)'-(N+1)/2, 1, K) .* repmat(cos(phi(:)'), N, 1));

end
```

Example: Uniform Circular Array



- radius:

$$r \sin\left(\frac{\pi}{N}\right) = \frac{d}{2}$$

$$\Rightarrow \quad r = \frac{d}{2\sin\left(\frac{\pi}{N}\right)}$$

- even number N of antenna elements for a centrosymmetric array

- rearrange antenna elements to obtain a conjugate centrosymmetric array manifold vector:

$$m = \begin{cases} n-1 & n \leq \frac{N}{2} \\ \frac{3N}{2} - n & n > \frac{N}{2} \end{cases}$$

- enclosed angle: $\varphi_k - \frac{2\pi}{N} m$ (for 2D)

- steering factor:

$$\underline{a}_{m,k} = e^{j\langle \vec{r}_m, \vec{\beta}_k \rangle}$$

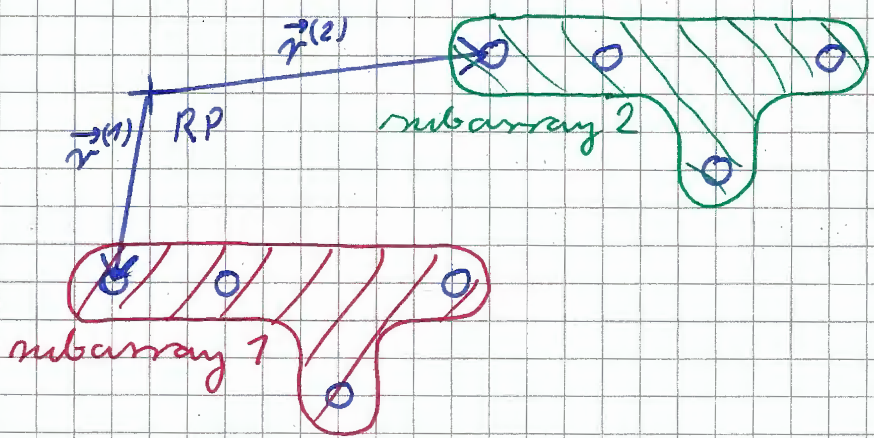
$$= e^{j r \beta \cos\left(\varphi_k - \frac{2\pi}{N} m\right)} \quad \text{(for 2D)}$$

```
function A = uca(phi, N, d)
%UCA compute the array manifold matrix of a uniform circular array
%    A: N x K array manifold matrix
%    phi: vector with K directions of arrival
%    N: even number of antenna elements
%    d: distance of antenna elements normalized to the wavelength

if rem(N, 2)~=0
    error('odd number of antenna elements');
end

r = d/(2*sin(pi/N)); % normailzed radius

K = numel(phi); % number of sources

% rearrange antenna elements
m = [(1:N/2)'-1; 3*N/2-(N/2+1:N)'];

% the steering factors are exp(j*2*pi*r*cos(phi-2*pi*m/N))
A = exp(j*2*pi*r*cos(repmat(phi(:)', N, 1)-2*pi*repmat(m, 1, K)/N));

end
```

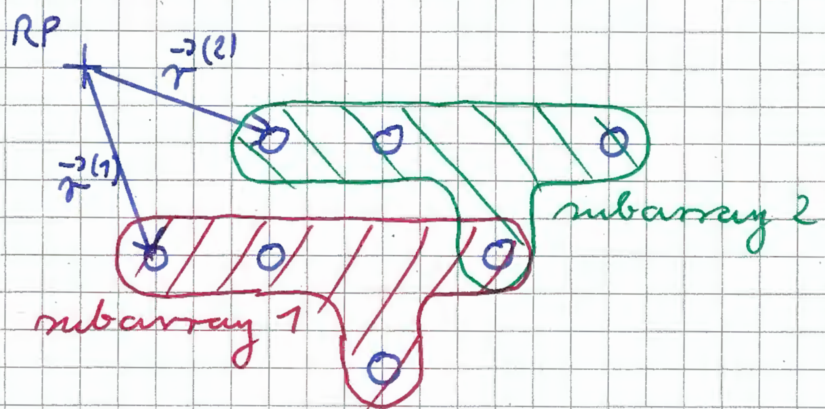
Subarrays

identical subarrays

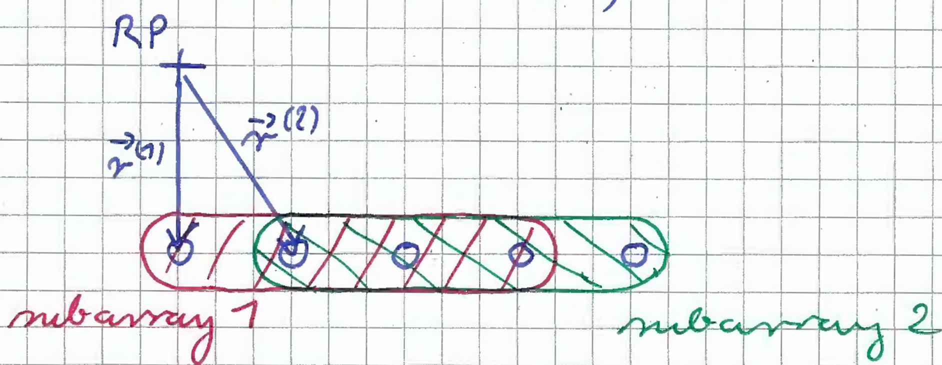
1.) two disjoint subarrays



2.) two partially overlapping subarrays



3.) two maximally overlapping subarrays (differ only in a single antenna element)



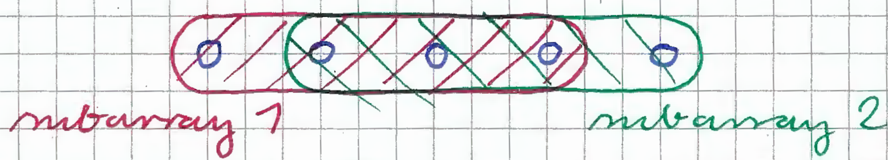
$\Rightarrow$ displacements $\vec{r}^{(m)}$

Selection Matrix

- With a properly chosen selection matrix $S^{(m)}$ and the array manifold matrix $\underline{A}$ (of the complete array) one obtains the array manifold matrix of the subarray

$$\underline{A}^{(m)} = S^{(m)} \cdot \underline{A} \ .$$

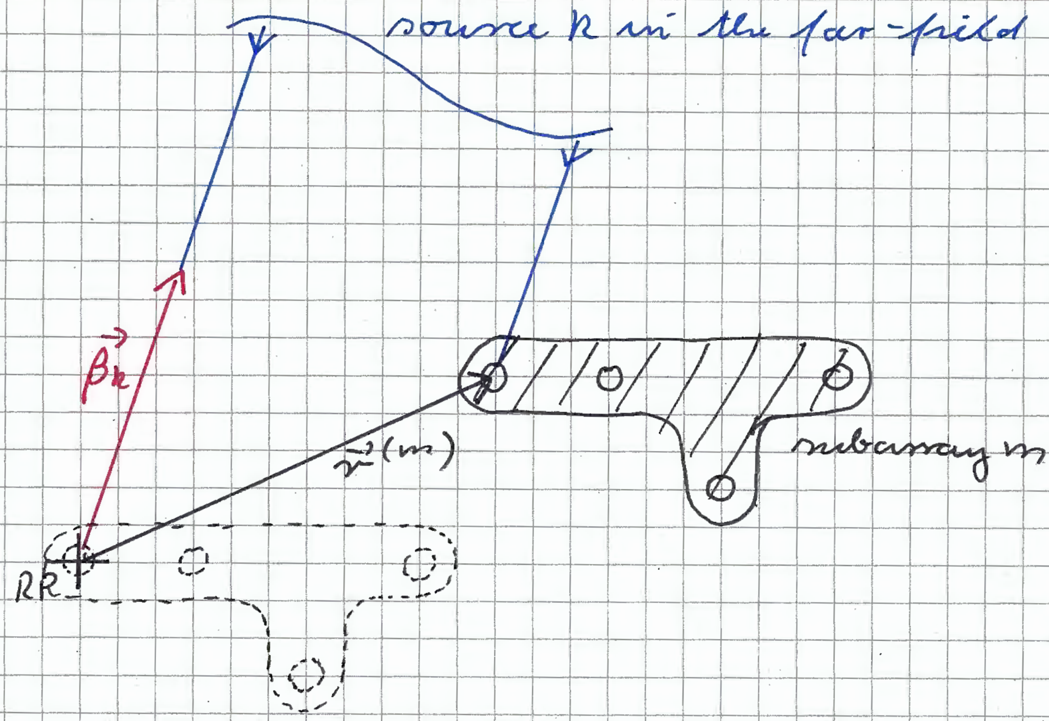
- example:
  two maximally overlapping subarrays



$$S^{(1)} = (E, 0)$$

$$S^{(2)} = (0, E)$$

Shift Invariance Property



- array manifold vector of the virtual subarray at the reference point : $\underline{a}_{RP,k}$

- array manifold vector of the m-th subarray :

$$\underline{a}_k^{(m)} = \underline{a}_{RP,k} e^{j\langle \vec{r}^{(m)}, \vec{\beta}_k \rangle}$$

- array manifold matrix of the m-th subarray :

$$\underline{A}^{(m)} = \underline{A}_{RP} \cdot \underbrace{\begin{pmatrix} e^{j\langle \vec{r}^{(m)}, \vec{\beta}_1 \rangle} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & e^{j\langle \vec{r}^{(m)}, \vec{\beta}_K \rangle} \end{pmatrix}}_{\underline{\phi}^{(m)}}$$

$\underline{\phi}^{(m)}$ is unitary

Noise Free Received Signal

linear superposition of the contributions of the K sources

- noise free received signal at the n-th antenna element:

$$\underline{s}_n = \sum_k \underline{s}_{n,k} = \sum_k \underline{a}_{n,k} \underbrace{\underline{s}_{RP,k}}_{\substack{\text{source signal} \\ \text{(virtual) received signal at} \\ \text{the reference point}}}$$

- noise free received vector, N antenna elements:

$$\underline{s} = \sum_k \underline{a}_k \, \underline{s}_{RP,k}$$

$$= \underline{A} \cdot \underbrace{\begin{pmatrix} \underline{s}_{RP,1} \\ \vdots \\ \underline{s}_{RP,K} \end{pmatrix}}_{\substack{\text{source vector } \underline{s}_{RP} \\ \text{(virtual) received vector} \\ \text{at the reference point}}} = \underline{A} \cdot \underline{s}_{RP}$$

- The columns of the array manifold matrix $\underline{A}$, i.e., the array manifold vectors $\underline{a}_k$, $k = 1 \dots K$, span the signal subspace.

Received Signal

additive noise, e.g., noise of the receiver

- received signal at the n-th antenna element, K sources:

$$\underline{e}_n = \underbrace{\sum_k \underline{s}_{n,k}}_{\underline{s}_n} + \underbrace{\underline{n}_n}_{\text{noise}} = \underbrace{\sum_k \underline{a}_{n,k}\,\underline{s}_{RP,k}}_{\underline{s}_n} + \underline{n}_n$$

- received vector, N antenna elements:

$$\underline{e} = \underbrace{\underline{A}\cdot\underline{s}_{RP}}_{\underline{s}} + \underbrace{\begin{pmatrix}\underline{n}_1\\ \vdots\\ \underline{n}_N\end{pmatrix}}_{\text{noise vector } \underline{n}} = \underbrace{\underline{A}\cdot\underline{s}_{RP}}_{\underline{s}} + \underline{n}$$

Example: Uniform Linear Array

- noise free received vector, single source k:

$$\underline{r} = \begin{pmatrix} \underline{\varepsilon}_k^{-\frac{N-1}{2}} \\ \underline{\varepsilon}_k^{-\frac{N-3}{2}} \\ \vdots \\ \underline{\varepsilon}_k^{+\frac{N-3}{2}} \\ \underline{\varepsilon}_k^{+\frac{N-1}{2}} \end{pmatrix} \underline{a}_{RP,k} = \begin{pmatrix} e^{-j\cdot\frac{N-1}{2} d\beta_{x,k}} \\ e^{-j\cdot\frac{N-3}{2} d\beta_{x,k}} \\ \vdots \\ e^{+j\cdot\frac{N-3}{2} d\beta_{x,k}} \\ e^{+j\cdot\frac{N-1}{2} d\beta_{x,k}} \end{pmatrix} \underline{a}_{RP,k}$$

- Direction of arrival estimation is a harmonic retrieval problem (estimate the "frequency" $\beta_{x,k}$).

- (Discrete) spatial domain (antenna element index n) and wavenumber domain (x-component β_x of the wavenumber vector $\vec{\beta}$) are a Fourier pair.

Aliasing

- Phases can only be measured up to integer multiples of 2π.

  $\Rightarrow \quad -\pi \leq d\,\beta_{x,k} \leq +\pi$

  for unambiguous measurements

- Using

  $$-\frac{2\pi}{\lambda} \leq \beta_{x,k} \leq +\frac{2\pi}{\lambda}$$

  one obtains

  $$d \leq \frac{\lambda}{2} .$$

  (spatial) sampling theorem

First Approach to Direction of Arrival Estimation

Measure the received signal in the spatial domain and compute the spectrum.

Challenges:

- sampling in the spatial domain
  $\Rightarrow$ avoid aliasing by sufficiently small antenna spacing $d \leq \frac{\lambda}{2}$

- finite number of antenna elements, finite aperture
  $\Rightarrow$ windowing in the spatial domain
  $\Rightarrow$ smoothing of the spectrum
  $\Rightarrow$ limited resolution
  $\Rightarrow$ Gibbs phenomenon

- noise
  $\Rightarrow$ averaging over many samples

- configurations different from the uniform linear array
  $\Rightarrow$ formulate algorithms as general as possible