Measurements

Sample Correlations

- L measured samples $\underline{e}_n^{(\ell)}$, $\ell = 1 \ldots L$, of the received signals

- sample correlations

$$\underline{\hat{r}}_{m,n} = \frac{1}{L} \sum_{\ell} \underline{e}_m^{(\ell)} \underline{e}_n^{(\ell)*}$$

  - estimates of correlations $\underline{r}_{m,n}$
  - $\underline{\hat{r}}_{n,m} = \underline{\hat{r}}_{m,n}^{*}$ still holds (if both are computed from the same set of samples)
  - even if the lags are the same the sample correlations will be different

Sample Correlation Matrix

- L measured samples $\underline{e}^{(\ell)}$, $\ell = 1 \dots L$, of the received vector
- data matrix

$$\underline{D} = \left(\underline{e}^{(1)}, \dots, \underline{e}^{(L)}\right) \qquad N \times L \text{ matrix}$$

- sample correlation matrix:

$$\hat{\underline{R}}_{ee} = \frac{1}{L} \sum_{\ell} \underline{e}^{(\ell)} \cdot \underline{e}^{(\ell)*T}$$

$$= \frac{1}{L} \underline{D} \cdot \underline{D}^{*T}$$

estimate of the correlation matrix $\underline{R}_{ee}$

- $\text{rank}(\hat{\underline{R}}_{ee}) = \min(N, L)$

$\Rightarrow$ at least $L = N$ samples to obtain a full rank sample correlation matrix

- $\hat{\underline{R}}_{ee}$ is always Hermitian:

$$\hat{\underline{R}}_{ee}^{*T} = \left(\frac{1}{L} \underline{D} \cdot \underline{D}^{*T}\right)^{*T}$$

$$= \frac{1}{L} \underline{D} \cdot \underline{D}^{*T}$$

$$= \hat{\underline{R}}_{ee}$$

- $\hat{\underline{R}}_{ee}$ is always positive semidefinite:

$$\underline{x}^{*T} \cdot \hat{\underline{R}}_{ee} \cdot \underline{x} = \underline{x}^{*T} \cdot \frac{1}{L} \cdot \underline{D} \cdot \underline{D}^{*T} \cdot \underline{x}$$

$$= \frac{1}{L} \left(\underline{x}^{*T} \cdot \underline{D}\right) \cdot \left(\underline{x}^{*T} \cdot \underline{D}\right)^{*T}$$

$$= \frac{1}{L} \| \underline{x}^{*T} \cdot \underline{D} \|^2$$

$$\geq 0 \text{ for all } \underline{x}$$

```
function Ree = correlation(A, RRP, sigma, L)
%CORRELATION determine the (sample) correlation matrix
%   Ree: correlation matrix
%   A: array manifold matrix
%   RRP: source correlation matrix
%   sigma: standard deviation of the noise
%   L: number of samples, zero means perfect averaging

N = size(A, 1); % number of antenna elements
K = size(A, 2); % number of sources

if L==0
    Ree = A*RRP*A'+sigma^2*eye(N); % correlation matrix
else
    % filter for generating correlated signals
    [V, D] = eig(RRP);

    % data matrix
    srp = V*sqrt(D)*sqrt(0.5)*(randn(K, L)+j*randn(K, L)); % source vector
    D = A*srp+sqrt(0.5)*sigma*(randn(N, L)+j*randn(N, L)); % noisy data matrix

    Ree = D*D'/L; % sample correlation matrix
end

end
```

Eigendecomposition

- singular value decomposition of the data matrix:

$$\underline{D} = \underline{\hat{U}} \cdot \underline{\hat{\Sigma}} \cdot \underline{\hat{V}}^{*T}$$

- eigendecomposition of the sample correlation matrix

$$\underline{\hat{R}}_{ee} = \frac{1}{L} \underline{D} \cdot \underline{D}^{*T}$$

$$= \frac{1}{L} \underline{\hat{U}} \cdot \underline{\hat{\Sigma}} \cdot \underline{V}^{*T} \cdot \underline{V} \cdot \underline{\hat{\Sigma}}^{T} \cdot \underline{U}^{*T}$$

$$= \underline{\hat{U}} \cdot \underbrace{\frac{1}{L} \cdot \underline{\hat{\Sigma}} \cdot \underline{\hat{\Sigma}}^{T}}_{\Lambda} \cdot \underline{U}^{*T}$$

$\Rightarrow$ The eigenvectors of $\underline{\hat{R}}_{ee}$ are the left singular vectors of $\underline{D}$.

Example: Sample Correlation Matrix

- uniform linear array:

$$\begin{array}{ccc} \circ & \circ & \circ \\ n=1 & 2 & 3=N \end{array}$$

exploiting the array geometry:

$$\underline{r}_{11} = \underline{r}_{22} = \underline{r}_{33}$$

$$\underline{r}_{12} = \underline{r}_{23} = \underline{r}_{21}^{*} = \underline{r}_{32}^{*}$$

$$\underline{r}_{13} = \underline{r}_{31}^{*}$$

- only a single sample $L = 1$:

$$\underline{e}^{(1)} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \hat{\underline{R}}_{ee} = \underline{e}^{(1)} \cdot \underline{e}^{(1)*T} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

- eigenvalues and corresponding eigenvectors:

$$\hat{\lambda}_1 = 6 \Rightarrow \hat{\underline{u}}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\left.\begin{array}{l} \hat{\lambda}_2 = 0 \Rightarrow \hat{\underline{u}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ \hat{\lambda}_3 = 0 \Rightarrow \hat{\underline{u}}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \end{array}\right\} \text{orthonormal complement for } \hat{\underline{u}}_1$$

$\hat{\underline{R}}_{ee}$ is rank deficient due to insufficient number of samples.

• <u>wrong</u> approach:

One might try to improve the result by averaging over the estimates of correlations which should be equal:

$$\Rightarrow \quad \hat{\underline{R}}_{ee} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

eigenvalues:

$$\hat{\lambda}_1 = \frac{5+\sqrt{33}}{2} \approx 5{,}372$$

$$\hat{\lambda}_2 = 1$$

$$\hat{\lambda}_3 = \frac{5-\sqrt{33}}{2} \approx -0{,}372$$

$\hat{\underline{R}}_{ee}$ is <u>not</u> a valid sample correlation matrix as it is not positive semidefinite.