Correlation Matrix

Correlation Matrices

- correlation matrix:

$$\underline{R}_{ss} = E\{\underline{s} \cdot \underline{s}^{*T}\}$$

- correlation matrices are Hermitian:

$$\underline{R}_{ss}^{*T} = \left(E\{\underline{s} \cdot \underline{s}^{*T}\}\right)^{*T}$$

$$= E\{\underline{s} \cdot \underline{s}^{*T}\}$$

$$= \underline{R}_{ss}$$

$\Rightarrow$ eigendecomposition exists, real eigenvalues

- correlation matrices are positive semidefinite:

$$\underline{x}^{*T} \cdot \underline{R}_{ss} \cdot \underline{x} = \underline{x}^{*T} \cdot E\{\underline{s} \cdot \underline{s}^{*T}\} \cdot \underline{x}$$

$$= E\{\underline{x}^{*T} \cdot \underline{s} \cdot \underline{s}^{*T} \cdot \underline{x}\}$$

$$= E\{|\underline{x}^{*T} \cdot \underline{s}|^2\}$$

$$\geq 0 \quad \text{for all } \underline{x}$$

$\Rightarrow$ nonnegative eigenvalues

- in the following eigenvalues in nonincreasing order:

$$\lambda_1 \geq \ldots \geq \lambda_R > \lambda_{R+1} = \ldots = \lambda_N = 0$$

Noise Free Correlation Matrix

- noise free correlation matrix:

$$\underline{R}_{ss} = E\{\underline{s} \cdot \underline{s}^{*T}\}$$

$$= E\{\underline{A} \cdot \underline{s}_{RP} \cdot \underline{s}_{RP}^{*T} \cdot \underline{A}^{*T}\}$$

$$= \underline{A} \cdot \underbrace{E\{\underline{s}_{RP} \cdot \underline{s}_{RP}^{*T}\}}_{\text{source correlation matrix } \underline{R}_{RP}} \cdot \underline{A}^{*T}$$

- in the following uncorrelated sources

$$\underline{R}_{RP} = \begin{pmatrix} \sigma_{RP,1}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{RP,K}^2 \end{pmatrix}$$

$$\Rightarrow \underline{R}_{ss} = \underline{A} \cdot \begin{pmatrix} \sigma_{RP,1}^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_{RP,K}^2 \end{pmatrix} \cdot \underline{A}^{*T}$$

$\Rightarrow$ both the columns of $\underline{A}$ and of $\underline{R}_{ss}$ span the same signal subspace

- rank:

$$R = \text{rank}(\underline{R}_{ss}) = \min(N, K)$$

in the following $N > K$
(more antenna elements than sources)
$\Rightarrow \underline{R}_{ss}$ is rank deficient

- eigendecomposition:

$$\underline{R}_{nn} = (\underline{u}_1, \ldots, \underline{u}_K, \underline{u}_{K+1}, \ldots, \underline{u}_N) \cdot \begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_K & & & \\ & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} \underline{u}_1^{*T} \\ \vdots \\ \underline{u}_K^{*T} \\ \underline{u}_{K+1}^{*T} \\ \vdots \\ \underline{u}_N^{*T} \end{pmatrix}$$

- The eigenvectors $\underline{u}_1, \ldots, \underline{u}_K$ corresponding to the nonzero eigenvalues $\lambda_1, \ldots, \lambda_K$ span the signal subspace.

- There is in general no one-to-one relation between sources (array manifold vectors) and eigenvectors.

Noise Free Received Power

total received power, i.e., sum of the received powers of all antenna elements:

$$P_s = E\{\|\underline{s}\|^2\}$$

$$= \operatorname{trace}(\underline{R}_{ss})$$

$$= \sum_n \lambda_n$$

Noise

- correlation matrix:

$$\underline{R}_{nn} = E\{\underline{n} \cdot \underline{n}^{*T}\}$$

- white noise:

$$\underline{R}_{nn} = \sigma^2 E$$

$\Rightarrow$ $\underline{R}_{nn}$ has full rank N

- single eigenvalue σ^2 with multiplicity N

$\Rightarrow$ one can choose any orthonormal basis of the N dimensional space as the normalized eigenvectors

- total noise power

$$P_n = E\{\|\underline{n}\|^2\}$$

$$= \operatorname{trace}(\underline{R}_{nn})$$

$$= N\sigma^2$$

Correlation Matrix

- correlation matrix:

$$\underline{R}_{ee} = E\{\underline{e}\cdot\underline{e}^{*T}\}$$
$$= E\{(\underline{s}+\underline{n})\cdot(\underline{s}^{*T}+\underline{n}^{*T})\}$$
$$= \underbrace{E\{\underline{s}\cdot\underline{s}^{*T}\}}_{\underline{R}_{ss}} + \underbrace{E\{\underline{s}\cdot\underline{n}^{*T}\}}_{0} + \underbrace{E\{\underline{n}\cdot\underline{s}^{*T}\}}_{0} + \underbrace{E\{\underline{n}\cdot\underline{n}^{*T}\}}_{\underline{R}_{nn}}$$
$$= \underline{R}_{ss} + \underline{R}_{nn} = \underline{R}_{ss} + \sigma^2 E$$

- elements:

$$[\underline{R}_{ee}]_{m,n} = r_{m,n}$$

- $\underline{R}_{ee}$ has the same eigenvectors $\underline{u}_n$ as $\underline{R}_{ss}$:

$$\underline{R}_{ee}\cdot\underline{u}_n = (\underline{R}_{ss}+\sigma^2 E)\cdot\underline{u}_n$$
$$= \underbrace{\underline{R}_{ss}\cdot\underline{u}_n}_{\lambda_n\underline{u}_n} + \underbrace{\sigma^2 E\cdot\underline{u}_n}_{\sigma^2\underline{u}_n}$$
$$= (\lambda_n+\sigma^2)\,\underline{u}_n$$

$\Rightarrow$ the eigenvalues are shifted by σ^2

notation: λ_n are the eigenvalues of $\underline{R}_{ss}$

- $\underline{R}_{ee}$ has full rank N

Eigendecomposition

eigendecomposition of the correlation matrix

$$\underline{R}_{ee} = \underbrace{(\underline{u}_1, \dots, \underline{u}_K, \underline{u}_{K+1}, \dots, \underline{u}_N)}_{\underline{U} = (\underline{U}_s, \underline{U}_n)} \cdot \underbrace{\begin{pmatrix} \lambda_1+\sigma^2 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_K+\sigma^2 & & & \\ & & & \sigma^2 & & \\ & & & & \ddots & \\ 0 & & & & & \sigma^2 \end{pmatrix}}_{\Lambda = \begin{pmatrix} \Lambda_s & 0 \\ 0 & \Lambda_n \end{pmatrix}} \cdot \underbrace{\begin{pmatrix} \underline{u}_1^{*T} \\ \vdots \\ \underline{u}_K^{*T} \\ \underline{u}_{K+1}^{*T} \\ \vdots \\ \underline{u}_N^{*T} \end{pmatrix}}_{\underline{U}^{*T} = \begin{pmatrix} \underline{U}_s^{*T} \\ \underline{U}_n^{*T} \end{pmatrix}}$$

$$= \underbrace{(\underline{u}_1, \dots, \underline{u}_K)}_{\underline{U}_s} \cdot \underbrace{\begin{pmatrix} \lambda_1+\sigma^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_K+\sigma^2 \end{pmatrix}}_{\Lambda_s} \cdot \underbrace{\begin{pmatrix} \underline{u}_1^{*T} \\ \vdots \\ \underline{u}_K^{*T} \end{pmatrix}}_{\underline{U}_s^{*T}}$$

$$+ \underbrace{(\underline{u}_{K+1}, \dots, \underline{u}_N)}_{\underline{U}_n} \cdot \underbrace{\begin{pmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{pmatrix}}_{\Lambda_n} \cdot \underbrace{\begin{pmatrix} \underline{u}_{K+1}^{*T} \\ \vdots \\ \underline{u}_N^{*T} \end{pmatrix}}_{\underline{U}_n^{*T}}$$

Two Subspaces

- The first K largest eigenvalues $\lambda_1+\sigma^2, \ldots, \lambda_K+\sigma^2$ are the principal eigenvalues.
  The corresponding eigenvectors $\underline{u}_1, \ldots, \underline{u}_K$ are the principal eigenvectors.

- $\underline{U}_s$, i.e., the principal eigenvectors $\underline{u}_1, \ldots, \underline{u}_K$ span the signal subspace of dimension K.

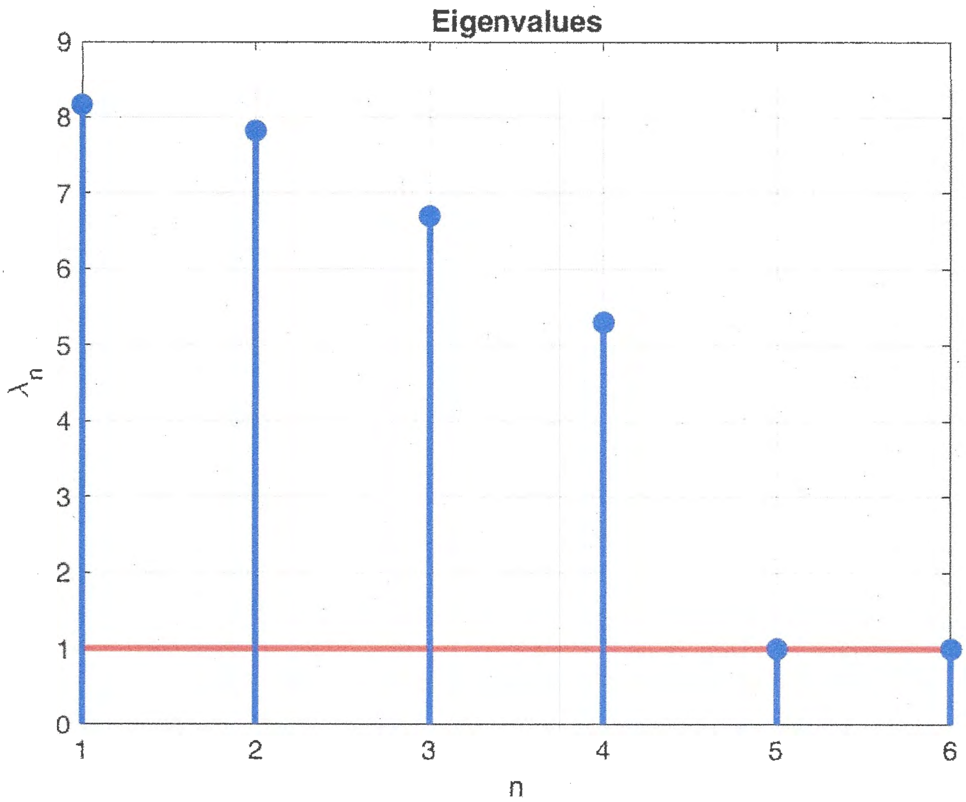
- The remaining $N-K$ eigenvalues (which are all σ^2) are the non principal eigenvalues.
  The corresponding eigenvectors $\underline{u}_{K+1}, \ldots, \underline{u}_N$ are the non principal eigenvectors

- $\underline{U}_n$, i.e., the non principal eigenvectors $\underline{u}_{K+1}, \ldots, \underline{u}_N$ span the noise subspace of dimension $N-K$.

$N = 6$

$K = 4$ sources with equal power

$\sigma^2 = 1$



Received Power

total received power including noise

$$P = E\{\|\underline{e}\|^2\}$$

$$= \mathrm{trace}(\underline{R}_{ee})$$

$$= \sum_n (\lambda_n + \sigma^2)$$

$$= \sum_n \lambda_n + N\sigma^2$$

$$= P_s + P_n$$

Source Correlation Matrix Estimation

- prerequisites:
  - directions of arrival known
    $\Rightarrow$ array manifold matrix $\underline{A}$ known
  - correlation matrix $\underline{R}_{ee}$ known
  - noise powers σ^2 known

- correlation matrix:

$$\underline{R}_{ee} = \underline{A} \cdot \underline{R}_{RP} \cdot \underline{A}^{*T} + \sigma^2 E$$

- vectorization:

$$\begin{aligned} \mathrm{vec}(\underline{R}_{ee} - \sigma^2 E) &= \mathrm{vec}(\underline{A} \cdot \underline{R}_{RP} \cdot \underline{A}^{*T}) \\ &= (\underline{A}^* \otimes \underline{A}) \cdot \mathrm{vec}(\underline{R}_{RP}) \end{aligned}$$

Kronecker product

- least squares pseudosolution:

$$\mathrm{vec}(\hat{\underline{R}}_{RP}) = (\underline{A}^* \otimes \underline{A})^+ \cdot \mathrm{vec}(\underline{R}_{ee} - \sigma^2 E)$$

• left pseudoinverse:

$$(\underline{A}^* \otimes \underline{A})^+ = \left((\underline{A}^* \otimes \underline{A})^{*T} \cdot (\underline{A}^* \otimes \underline{A})\right)^{-1} \cdot (\underline{A}^* \otimes \underline{A})^{*T}$$

$$= \left((\underline{A}^T \otimes \underline{A}^{*T}) \cdot (\underline{A}^* \otimes \underline{A})\right)^{-1} \cdot (\underline{A}^T \otimes \underline{A}^{*T})$$

$$= \left((\underline{A}^T \cdot \underline{A}^*) \otimes (\underline{A}^{*T} \cdot \underline{A})\right)^{-1} \cdot (\underline{A}^T \otimes \underline{A}^{*T})$$

$$= \left((\underline{A}^T \cdot \underline{A}^*)^{-1} \otimes (\underline{A}^{*T} \cdot \underline{A})^{-1}\right) \cdot (\underline{A}^T \otimes \underline{A}^{*T})$$

$$= \left((\underline{A}^T \cdot \underline{A})^{-1} \cdot \underline{A}^T\right) \otimes \left((\underline{A}^{*T} \cdot \underline{A})^{-1} \cdot \underline{A}^{*T}\right)$$

$$= (\underline{A}^+)^* \otimes \underline{A}^+$$

• source correlation matrix:

$$\mathrm{vec}(\hat{\underline{R}}_{RP}) = \left((\underline{A}^+)^* \otimes \underline{A}^+\right) \cdot \mathrm{vec}(\underline{R}_{ee} - \sigma^2 E)$$

$$= \mathrm{vec}\left(\underline{A}^+ \cdot (\underline{R}_{ee} - \sigma^2 E) \cdot (\underline{A}^+)^{*T}\right)$$

$$\Rightarrow \quad \hat{\underline{R}}_{RP} = \underline{A}^+ \cdot (\underline{R}_{ee} - \sigma^2 E) \cdot (\underline{A}^+)^{*T}$$

• reconstructed correlation matrix:

$$\hat{\underline{R}}_{ee} = \underline{A} \cdot \hat{\underline{R}}_{RP} \cdot \underline{A}^{*T} + \sigma^2 E$$

$$= \underbrace{\underline{A} \cdot \underline{A}^+}_{\underline{P}} \cdot (\underline{R}_{ee} - \sigma^2 E) \cdot \underbrace{(\underline{A}^+)^{*T} \cdot \underline{A}^{*T}}_{\underline{P}^{*T}} + \sigma^2 E$$

Projection Matrix

- Using the singular value decomposition

$$\underline{A} = \underline{U} \cdot \Sigma \cdot \underline{V}^{*T} \quad , \; N \times K \text{ matrix}$$

one obtains:

$$\underline{P} = \underline{A} \cdot \underline{A}^{+}$$

$$= \underline{A} \cdot \underbrace{(\underline{A}^{*T} \cdot \underline{A})^{-1} \cdot \underline{A}^{*T}}_{\text{left pseudoinverse}}$$

$$= \underline{U} \cdot \Sigma \cdot \underline{V}^{*T} \cdot \underline{V} \cdot \Sigma^{+} \cdot \underline{U}^{*T}$$

$$= \underline{U} \cdot \underbrace{\Sigma \cdot \Sigma^{+}}_{\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}} \cdot \underline{U}^{*T}$$

$K \times K$

$$= (\underline{u}_1, \ldots, \underline{u}_K) \cdot \begin{pmatrix} \underline{u}_1^{*T} \\ \vdots \\ \underline{u}_K^{*T} \end{pmatrix}$$

$\Rightarrow$ projection on the K dimensional column space of $\underline{A}$, i.e., on the signal subspace

Properties of the Projection Matrix

- $\underline{P} = \underline{A} \cdot (\underline{A}^{*T} \cdot \underline{A})^{-1} \cdot \underline{A}^{*T} = \underline{P}^{*T}$

- $\underline{P} \cdot \underline{P} = \underline{A} \cdot (\underline{A}^{*T} \cdot \underline{A})^{-1} \cdot \underline{A}^{*T} \cdot \underline{A} \cdot (\underline{A}^{*T} \cdot \underline{A})^{-1} \cdot \underline{A}^{*T} = \underline{P}$

- $\underline{P}^{*T} \cdot \underline{P} = \underline{P} \cdot \underline{P} = \underline{P}$

- $$\begin{aligned} \mathrm{trace}(\underline{P}) &= \mathrm{trace}(\underline{A} \cdot (\underline{A}^{*T} \cdot \underline{A})^{-1} \cdot \underline{A}^{*T}) \\ &= \mathrm{trace}(\underline{A}^{*T} \cdot \underline{A} \cdot (\underline{A}^{*T} \cdot \underline{A})^{-1}) \\ &= \mathrm{trace}(E) \\ &= K \end{aligned}$$

```
function RRPe = sourcecorrelation(A, Ree, sigma)
%SOURCECORRELATION estimate the source correlation matrix
%   RRPe: source correlation matrix
%   A: array manifold matrix
%   Ree: correlation matrix
%   sigma: standard deviation of the noise

N = size(A, 1); % number of antenna elements
K = size(A, 2); % number of sources

RRPe = pinv(A)*(Ree-sigma^2*eye(N))*pinv(A)';

end
```

Orthogonal Projection Matrix

$$\underline{P}^{\perp} = E - \underline{P}$$

$$= \underline{U} \cdot \underline{U}^{*T} - (\underline{u}_1, \ldots, \underline{u}_K) \cdot \begin{pmatrix} \underline{u}_1^{*T} \\ \vdots \\ \underline{u}_K^{*T} \end{pmatrix}$$

$$= \underline{U} \cdot \begin{pmatrix} 0 & & & & & 0 \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix} \cdot \underline{U}^{*T}$$

$$= (\underline{u}_{K+1}, \ldots, \underline{u}_N) \cdot \begin{pmatrix} \underline{u}_{K+1}^{*T} \\ \vdots \\ \underline{u}_N^{*T} \end{pmatrix}$$

$\Rightarrow$ projection on the $N-K$ dimensional noise subspace

- properties:

$$\underline{P}^{\perp} = (\underline{P}^{\perp})^{*T}$$

$$\underline{P}^{\perp} \cdot \underline{P}^{\perp} = \underline{P}^{\perp}$$

$$(\underline{P}^{\perp})^{*T} \cdot \underline{P}^{\perp} = \underline{P}^{\perp}$$

$$\mathrm{trace}(\underline{P}^{\perp}) = N-K$$

Noise Power Estimation

- prerequisites:
  - directions of arrival known
    $\Rightarrow$ array manifold matrix $\underline{A}$ known
  - correlation matrix $\underline{R}_{ee}$ known

- idea:
  projection of $\underline{R}_{ee}$ on the
  noise subspace

$$\mathrm{trace}(\underline{P}^{\perp}\cdot \underline{R}_{ee}) = \mathrm{trace}\left((\underline{u}_{K+1},\dots,\underline{u}_N)\cdot \begin{pmatrix} \underline{u}_{K+1}^{*T} \\ \vdots \\ \underline{u}_N^{*T} \end{pmatrix} \cdot \underline{U} \cdot \begin{pmatrix} \lambda_1+\sigma^2 & & & & 0 \\ & \ddots & & & \\ & & \lambda_K+\sigma^2 & & \\ & & & \sigma^2 & \\ & & & & \ddots \\ 0 & & & & & \sigma^2 \end{pmatrix} \cdot \underline{U}^{*T} \right)$$

$$= \mathrm{trace}\left(\underbrace{\begin{pmatrix} \underline{u}_{K+1}^{*T} \\ \vdots \\ \underline{u}_N^{*T} \end{pmatrix} \cdot \underline{U}}_{\begin{pmatrix} 0 & E \end{pmatrix}} \cdot \begin{pmatrix} \lambda_1+\sigma^2 & & & & 0 \\ & \ddots & & & \\ & & \lambda_K+\sigma^2 & & \\ & & & \sigma^2 & \\ & & & & \ddots \\ 0 & & & & & \sigma^2 \end{pmatrix} \cdot \underbrace{\underline{U}^{*T}\cdot (\underline{u}_{K+1},\dots,\underline{u}_N)}_{\begin{pmatrix} 0 \\ E \end{pmatrix}} \right) = (N-K)\,\sigma^2$$

$N-K\times K$ $\quad$ $N-K\times N-K$

$$\Rightarrow \quad \sigma^2 = \frac{\mathrm{trace}(\underline{P}^{\perp}\cdot \underline{R}_{ee})}{N-K}$$