Maximum Likelihood Estimator

Parameter Estimation

- parameter vector (e.g. directions of arrival):

$$\theta = \begin{pmatrix} \theta_1 \\ \\ \theta_K \end{pmatrix}$$

- observation vector (e.g. received vector):

$$\underline{e} = \begin{pmatrix} \underline{e}_1 \\ \vdots \\ \underline{e}_N \end{pmatrix}$$

- system model (likelihood function):

$$p(\underline{e} \mid \theta)$$

- estimation problem:
  determine an estimate $\hat{\theta}$ of the parameter vector θ based on the observation vector $\underline{e}$

Maximum Likelihood Estimator

- maximum likelihood estimator:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \left\{ p(\underline{e} \mid \theta) \right\}$$

- additive noise:

$$\underline{e} = f(\theta) + \underline{n}$$

$\underline{n}$ ↳ noise

$f(\theta)$ ↳ (nonlinear) function

$$\Rightarrow \hat{\theta} = \underset{\theta}{\operatorname{argmax}} \left\{ p(\underline{n} = \underline{e} - f(\theta)) \right\}$$

- special case white Gaussian noise:

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} \left\{ \frac{1}{\pi^N \sigma^{2N}} e^{-\frac{\| \underline{e} - f(\theta) \|^2}{\sigma^2}} \right\}$$

$$= \underset{\theta}{\operatorname{argmin}} \left\{ \| \underline{e} - f(\theta) \|^2 \right\}$$

$\Rightarrow$ nonlinear least squares (NLS)

- linear function:

$$f(\theta) = \underline{F} \cdot \theta$$

$\underline{F}$ ↳ matrix

$$\Rightarrow \hat{\theta} = \underset{\theta}{\operatorname{argmin}} \left\{ \| \underline{e} - \underline{F} \cdot \theta \|^2 \right\}$$

$$= \underline{F}^{+} \cdot \underline{e}$$

$\underline{F}^{+}$ ↳ pseudoinverse

$\Rightarrow$ (linear) least squares (LS)

Example: Source Signal Estimation with Known Directions of Arrival

- known directions of arrival

  $\Rightarrow$ known array manifold matrix $\underline{A}$

- linear system model

$$\underline{e} = \underline{A} \cdot \underline{s}_{RP} + \underline{n}$$

- least squares estimate

$$\hat{\underline{s}}_{RP} = \underline{A}^{+} \cdot \underline{e}$$