Deterministic Maximum Likelihood (DML)

System Model

- directions of arrival:

$$\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_K \end{pmatrix}$$

- received vector, ℓ-th sample:

$$\underline{e}^{(\ell)} = \underline{A}(\varphi) \cdot \underline{s}_{RP}^{(\ell)} + \underline{n}^{(\ell)}$$

- data matrix, L samples:

$$\underbrace{\left(\underline{e}^{(1)}, \dots, \underline{e}^{(L)}\right)}_{\underline{D}} = \underline{A}(\varphi) \cdot \underbrace{\left(\underline{s}_{RP}^{(1)}, \dots, \underline{s}_{RP}^{(L)}\right)}_{\underline{S}_{RP}} + \underbrace{\left(\underline{n}^{(1)}, \dots, \underline{n}^{(L)}\right)}_{\underline{N}}$$

- white Gaussian noise:

$$p(\underline{D} \mid \underline{A}(\varphi), \underline{S}_{RP}) = \prod_{\ell} \frac{1}{\pi^N \sigma^{2N}} e^{-\frac{\| \underline{e}^{(\ell)} - \underline{A}(\varphi) \cdot \underline{s}_{RP}^{(\ell)} \|^2}{\sigma^2}}$$

$$= \frac{1}{(\pi \sigma^2)^{NL}} e^{-\frac{\| \underline{D} - \underline{A}(\varphi) \cdot \underline{S}_{RP} \|_F^2}{\sigma^2}}$$

Deterministic Maximum Likelihood (DML)

- deterministic = source signals $\underline{S}_{RP}$ as a parameter

- white gaussian noise
  ⇒ least squares estimator:

$$\hat{\varphi} = \underset{\varphi,\, \underline{S}_{RP}}{\operatorname{argmin}} \left\{ \| \underline{D} - \underline{A}(\varphi) \cdot \underline{S}_{RP} \|_F^2 \right\}$$

- assuming φ to be known
  ⇒ linear least squares estimate of the source signals:

$$\underline{S}_{RP} = \underline{A}^+(\varphi) \cdot \underline{D}$$

$$\Rightarrow \hat{\varphi} = \underset{\varphi}{\operatorname{argmin}} \left\{ \| \underline{D} - \underbrace{\underline{A}(\varphi) \cdot \underline{A}^+(\varphi)}_{\substack{\underline{P}(\varphi) \\ \text{projection matrix}}} \cdot \underline{D} \|_F^2 \right\}$$

Projection Matrix

- Using the singular value decomposition

$$\underline{A}(\varphi) = \underline{U} \cdot \Sigma \cdot \underline{V}^{*T} \quad , \; N \times K \text{ matrix}$$

one obtains:

$$P(\varphi) = \underline{A}(\varphi) \cdot \underline{A}^{+}(\varphi)$$

$$= \underline{U} \cdot \Sigma \cdot \underline{V}^{*T} \cdot \underline{V} \cdot \Sigma^{+} \cdot \underline{U}^{*T}$$

$$= \underline{U} \; \underbrace{\Sigma \cdot \Sigma^{+}}_{K \times K \;\to\; \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}} \cdot \underline{U}^{*T}$$

$$= (\underline{u}_1, \dots, \underline{u}_K) \cdot \begin{pmatrix} \underline{u}_1^{*T} \\ \vdots \\ \underline{u}_K^{*T} \end{pmatrix}$$

$\Rightarrow$ projection on the K dimensional column subspace of $\underline{A}(\varphi)$, i.e., the signal subspace

- properties:

$$\left.\begin{aligned} \underline{P}(\varphi) &= \underline{P}^{*T}(\varphi) \\ \underline{P}(\varphi) &= \underline{P}(\varphi) \cdot \underline{P}(\varphi) \end{aligned}\right\} \; \underline{P}^{*T}(\varphi) \cdot \underline{P}(\varphi) = \underline{P}(\varphi)$$

Nonlinear Least Squares Estimator

$$\hat{\varphi} = \underset{\varphi}{\operatorname{argmin}} \left\{ \| \underline{D} - \underline{P}(\varphi) \cdot \underline{D} \|_F^2 \right\}$$

$$= \underset{\varphi}{\operatorname{argmin}} \left\{ \operatorname{trace} \left((\underline{D} - \underline{P}(\varphi) \cdot \underline{D})^{*T} \cdot (\underline{D} - \underline{P}(\varphi) \cdot \underline{D}) \right) \right\}$$

$$= \underset{\varphi}{\operatorname{argmin}} \left\{ \operatorname{trace} \left(\underline{D}^{*T} \cdot \underline{D} - \underline{D}^{*T} \cdot \underline{P}^{*T}(\varphi) \cdot \underline{D} \right. \right.$$

$$\left. \left. - \underline{D}^{*T} \cdot \underline{P}(\varphi) \cdot \underline{D} + \underline{D}^{*T} \cdot \underline{P}^{*T}(\varphi) \cdot \underline{P}(\varphi) \cdot \underline{D} \right) \right\}$$

$$= \underset{\varphi}{\operatorname{argmax}} \left\{ \operatorname{trace} \left(\underline{D}^{*T} \cdot \underline{P}(\varphi) \cdot \underline{D} \right) \right\}$$

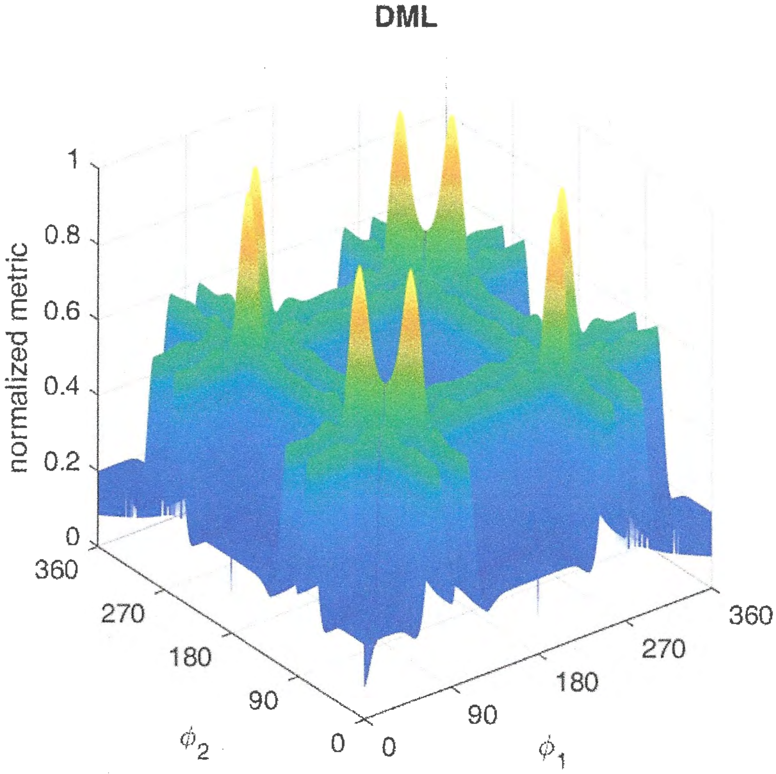
$$= \underset{\varphi}{\operatorname{argmax}} \left\{ \operatorname{trace} \left(\underline{P}(\varphi) \cdot \underline{D} \cdot \underline{D}^{*T} \right) \right\}$$

$$= \underset{\varphi}{\operatorname{argmax}} \left\{ \operatorname{trace} \left(\underline{P}(\varphi) \cdot \hat{\underline{R}}_{ee} \right) \right\}$$

- no closed form solution
- All the relevant information about the received signals is contained in the sample correlation matrix $\hat{\underline{R}}_{ee}$.

  $\Rightarrow$ $\hat{\underline{R}}_{ee}$ is a set of sufficient statistics

ULA
$N=6$
$d=0{,}5\lambda$
$\sigma^2=1$
$\varphi_1=75°$
$\varphi_2=105°$ } equal power



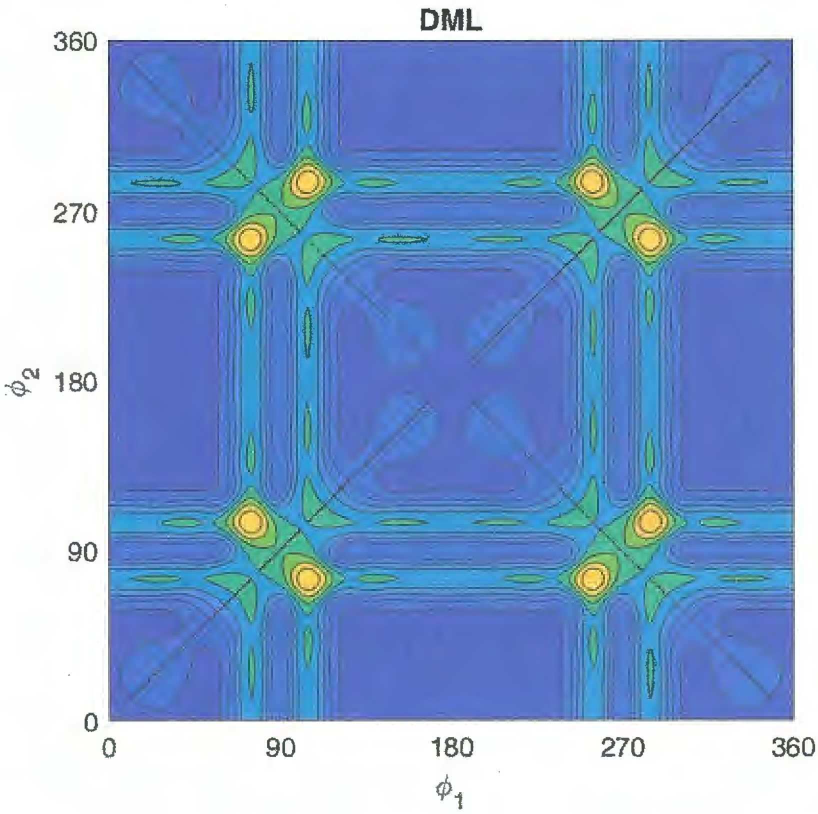
ULA

$N=6$

$d=0{,}5\lambda$

$\sigma^2=1$

$$\left.\begin{array}{l}\varphi_1=75^\circ\\ \varphi_2=105^\circ\end{array}\right\}\text{ equal power } 1$$



```
function [phi, value] = dml(Ree, K, d, manifold, linear, sigma, varargin)
%DML deterministic maximum likelihood
%   phi: directions of arrival
%   value: metric
%   Ree: correlation matrix
%   K: number of sources
%   d: distance of antenna elements normalized to the wavelength
%   manifold: antenna array
%   linear: limit search range to pi for linear arrays
%   sigma: standard deviation of the noise
%   varargin: up to K fixed directions of arrival

samples = 1001; % number of samples for the grid search

N = length(Ree); % number of antenna elements

if linear
    limit = pi;
else
    limit = 2*pi;
end

if numel(cell2mat(varargin))==K % all directions of arrival fixed
    % evaluate the metric
    value = real(trace(manifold(cell2mat(varargin), N, d)...
        *pinv(manifold(cell2mat(varargin), N, d))*Ree));
    phi = cell2mat(varargin);
else
    value = -inf;
    for psi = linspace(0, 1, samples)*limit % candidate directions of arrival
        [testphi, testvalue] = dml(Ree, K, d, manifold, linear, sigma,...
            cell2mat(varargin), psi);
        if testvalue>value
            phi = testphi;
            value = testvalue;
        end
    end
end

end
```

Special Case: Single Source

- array manifold vector: $\underline{a}(\varphi)$
- projection matrix:

$$\underline{P}(\varphi) = \frac{\underline{a}(\varphi) \cdot \underline{a}^{*T}(\varphi)}{\underline{a}^{*T}(\varphi) \cdot \underline{a}(\varphi)}$$

- nonlinear least squares estimator:

$$\hat{\varphi} = \underset{\varphi}{\operatorname{argmax}} \left\{ \operatorname{trace}\left(\frac{\underline{a}(\varphi) \cdot \underline{a}^{*T}(\varphi)}{\underline{a}^{*T}(\varphi) \cdot \underline{a}(\varphi)} \cdot \hat{\underline{R}}_{ee} \right) \right\}$$

$$= \underset{\varphi}{\operatorname{argmax}} \left\{ \operatorname{trace}\left(\frac{\underline{a}^{*T}(\varphi) \cdot \hat{\underline{R}}_{ee} \cdot \underline{a}(\varphi)}{\underline{a}^{*T}(\varphi) \cdot \underline{a}(\varphi)} \right) \right\}$$

$$= \underset{\varphi}{\operatorname{argmax}} \left\{ \frac{\underline{a}^{*T}(\varphi) \cdot \hat{\underline{R}}_{ee} \cdot \underline{a}(\varphi)}{\underline{a}^{*T}(\varphi) \cdot \underline{a}(\varphi)} \right\}$$

$\Rightarrow$ conventional beamforming