Capon's Beamformer

(Minimum Variance Distortionless Response, MVDR)

Capon's Beamformer

- idea:

Design an adaptive beamformer for suppressing the noise and the interferences.

⇒ Minimize the power (variance)

$$E\{|\underline{w}^{*T} \cdot \underline{e}|^2\} = \underline{w}^{*T} \cdot \underline{R}_{ee} \cdot \underline{w}$$

at the output of the beamforming network under the constraint of a distortionless response

$$\underline{F}(\varphi) = \underline{w}^{*T} \cdot \underline{a}(\varphi) = 1$$

to a source with the direction of arrival φ resulting in the array manifold vector $\underline{a}(\varphi)$.

Remark: $\underline{w}^*$ will not be normalized

- Using the eigendecomposition

$$\underline{R}_{ee} = \underline{U} \cdot \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}}_{\Lambda} \cdot \underline{U}^{*T}$$

and the definition

$$\Lambda^{1/2} = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_N} \end{pmatrix}$$

the objective function can be rewritten as

$$\underline{w}^{*T} \cdot \underline{R}_{ee} \cdot \underline{w} = \underbrace{\underline{w}^{*T} \cdot \underline{U} \cdot \Lambda^{1/2}}_{\underline{v}^{*T}} \cdot \underbrace{\Lambda^{1/2} \cdot \underline{U}^{*T} \cdot \underline{w}}_{\underline{v}}$$

$$= \|\underline{v}\|^2 .$$

$\Rightarrow$ Find the minimum square norm $\|\underline{v}\|^2$ pseudosolution $\underline{v}$ of the underdetermined linear system of equations

$$1 = \underline{a}^{*T}(\varphi) \cdot \underline{w}$$

$$= \underbrace{\underline{a}^{*T}(\varphi) \cdot \underline{U} \cdot \Lambda^{-1/2}}_{\underline{A}} \cdot \underline{v}$$

- pseudosolution using the right pseudoinverse:

$$\underline{v} = \underline{A}^{*T} \cdot (\underline{A} \cdot \underline{A}^{*T})^{-1} \cdot 1$$

$$= \Lambda^{-1/2} \cdot \underline{U}^{*T} \cdot \underline{a}(\psi) \cdot \left(\underline{a}^{*T}(\psi) \cdot \underline{U} \cdot \Lambda^{-1/2} \cdot \Lambda^{-1/2} \cdot \underline{U}^{*T} \cdot \underline{a}(\psi) \right)^{-1}$$

$$= \Lambda^{-1/2} \cdot \underline{U}^{*T} \cdot \underline{a}(\psi) \cdot \left(\underline{a}^{*T}(\psi) \cdot \underbrace{\underline{U} \cdot \Lambda^{-1} \cdot \underline{U}^{*T}}_{\underline{R}_{ee}^{-1}} \cdot \underline{a}(\psi) \right)^{-1}$$

$$= \frac{\Lambda^{-1/2} \cdot \underline{U}^{*T} \cdot \underline{a}(\psi)}{\underline{a}^{*T}(\psi) \cdot \underline{R}_{ee}^{-1} \cdot \underline{a}(\psi)}$$

- (conjugate complex) weight vector:

$$\underline{w} = \underline{U} \cdot \Lambda^{-1/2} \cdot \frac{\Lambda^{-1/2} \cdot \underline{U}^{*T} \cdot \underline{a}(\psi)}{\underline{a}^{*T}(\psi) \cdot \underline{R}_{ee}^{-1} \cdot \underline{a}(\psi)}$$

$$= \frac{\overbrace{\underline{U} \cdot \Lambda^{-1} \cdot \underline{U}^{*T}}^{\underline{R}_{ee}^{-1}} \cdot \underline{a}(\psi)}{\underline{a}^{*T}(\psi) \cdot \underline{R}_{ee}^{-1} \cdot \underline{a}(\psi)}$$

$$= \frac{\underline{R}_{ee}^{-1} \cdot \underline{a}(\psi)}{\underline{a}^{*T}(\psi) \cdot \underline{R}_{ee}^{-1} \cdot \underline{a}(\psi)} \qquad \text{not normalized!}$$

- power at the output of the beamforming network:

$$P_{Capon}(\varphi) = \underline{w}^{*T} \cdot \underline{R}_{ee} \cdot \underline{w}$$

$$= \frac{\underline{a}^{*T}(\varphi) \cdot \underline{R}_{ee}^{-1} \cdot \underline{R}_{ee} \cdot \underline{R}_{ee}^{-1} \cdot \underline{a}(\varphi)}{\underline{a}^{*T}(\varphi) \cdot \underline{R}_{ee}^{-1} \cdot \underline{a}(\varphi) \cdot \underline{a}^{*T}(\varphi) \cdot \underline{R}_{ee}^{-1} \cdot \underline{a}(\varphi)}$$

$$= \frac{1}{\underline{a}^{*T}(\varphi) \cdot \underline{R}_{ee}^{-1} \cdot \underline{a}(\varphi)}$$

- beamscan algorithm:
  Search for the (local) maxima of the pseudospectrum $P_{Capon}(\varphi)$.

```
function phi = capon(Ree, K, d, manifold, linear, varargin)
%CAPON Capon's beamformer (minimum variance distortionless response, MVDR)
%   phi: directions of arrival
%   Ree: correlation matrix
%   K: number of sources
%   d: distance of antenna elements normalized to the wavelength
%   manifold: antenna array
%   linear: limit search range to pi for linear arrays

samples = 1001; % number of samples for plotting the pseudospectrum

N = length(Ree); % number of antenna elements

% Using the array manifold vector the pseudospectrum is computed as 1/(a'*inv(Ree)*a).
% Passing a vector of directions of arrival one obtains a
% vector of samples of the pseudospectrum.
pseudospectrum = @(phi) real(1./sum(conj(manifold(phi,N,d)).*(inv(Ree)*manifold(phi,N,
d)),1));

% determine all local maxima
phi = maxsearch(pseudospectrum, linear);

% select the K largest local maxima
K = min([K, numel(phi)]);
phi = phi(1:K);

% plot the normalized pseudospectrum
figure;
psi = linspace(0, 1, samples)*2*pi;
h = polar(psi, pseudospectrum(psi)./max(pseudospectrum(psi)), '-');
set(h, 'LineWidth', 2);
set(h, 'Color', 'b');
hold on;
for k = 1:K % plot the estimated directions of arrival
    h = polar([phi(k) phi(k)], [0 1], '-');
    set(h, 'LineWidth', 2);
    set(h, 'Color', 'r');
end
title('Pseudospectrum');

end
```

ULA

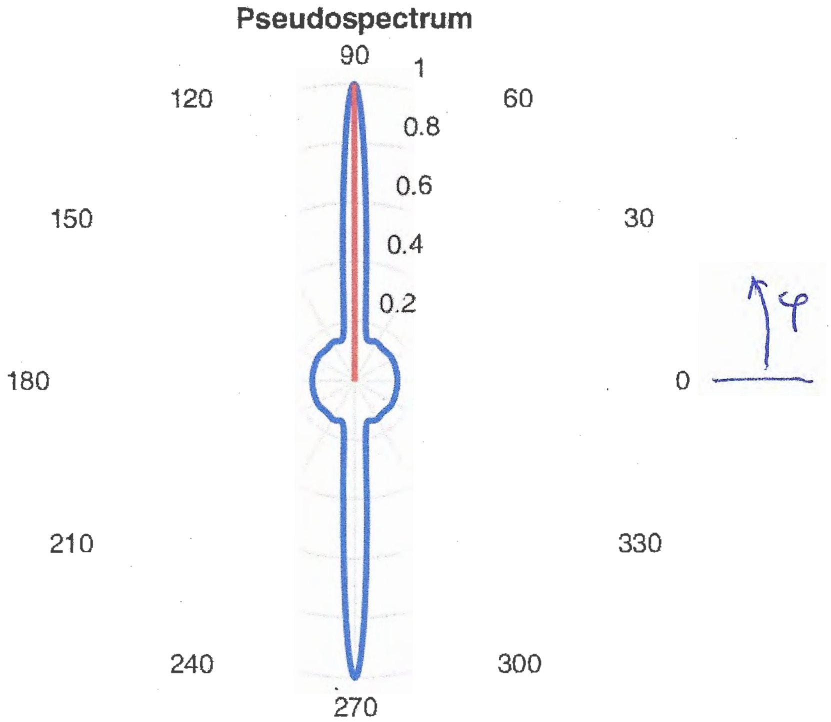
$N=6$

$d=0{,}5\lambda$

$\sigma^2=1$

$\varphi_1 = 90^\circ$ power 1

Capon



$\hat{\varphi}_1 = 90^\circ$

ULA

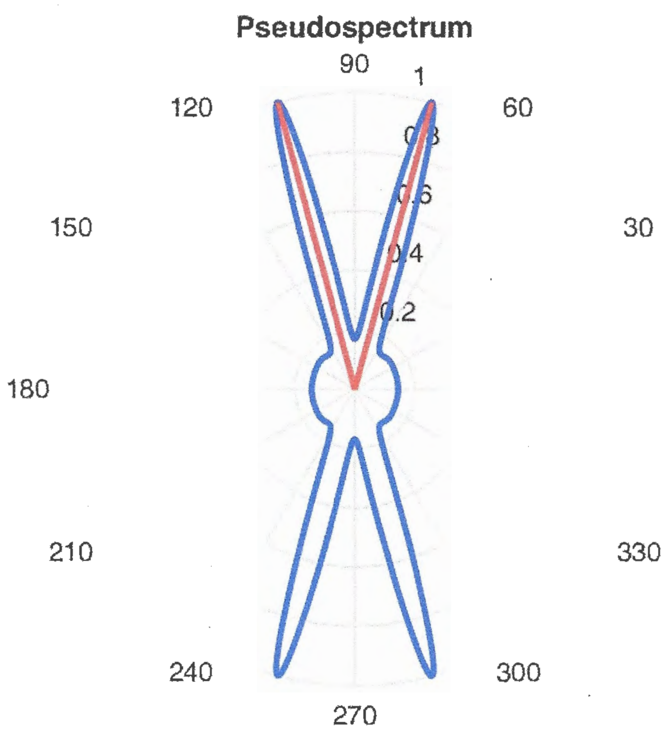
$N=6$

$d=0{,}5\lambda$

$\sigma^2=1$

$$\left.\begin{array}{l}\varphi_1 = 75^\circ \\ \varphi_2 = 105^\circ\end{array}\right\}\text{ equal power } 1$$

capon



$\tilde{\varphi}_1 = 104{,}95^\circ$

$\tilde{\varphi}_2 = 75{,}05^\circ$

ULA

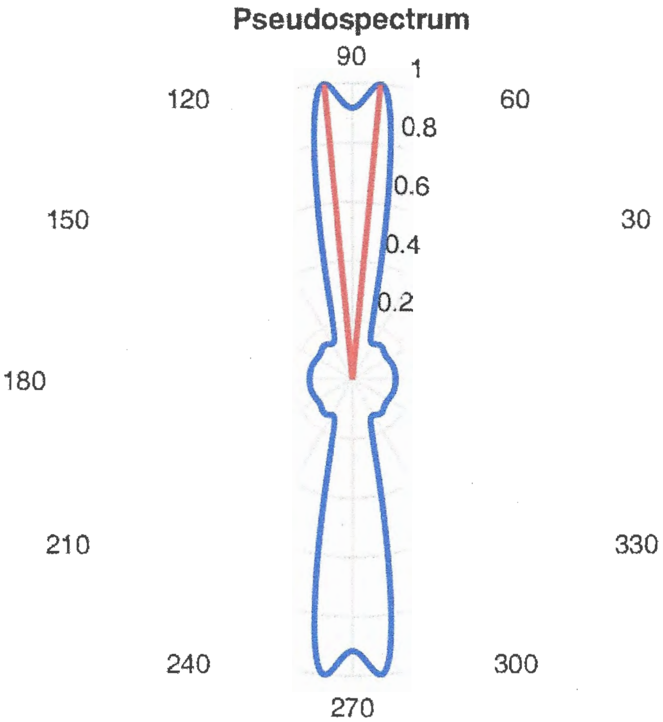
$N=6$

$d=0{,}5\lambda$

$\sigma^2=1$

$$\left.\begin{array}{l}\varphi_1=83^\circ\\ \varphi_2=97^\circ\end{array}\right\}\text{ equal power 1}$$

capon



$\hat{\varphi}_1 = 84{,}54^\circ$

$\hat{\varphi}_2 = 95{,}46^\circ$