Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT)

ESPRIT

- prerequisite:
  two identical subarrays

- invariance property:

$$\underline{A}^{(1)} = \underline{A}_{RP} \cdot \underline{\Phi}^{(1)}$$

$$\underline{A}^{(2)} = \underline{A}_{RP} \cdot \underline{\Phi}^{(2)}$$

$$\Rightarrow \underline{A}^{(2)} = \underline{A}^{(1)} \cdot \underbrace{\begin{pmatrix} e^{j\langle \vec{r}^{(2)} - \vec{r}^{(1)}, \vec{\beta}_1 \rangle} & & 0 \\ & \ddots & \\ 0 & & e^{j\langle \vec{r}^{(2)} - \vec{r}^{(1)}, \vec{\beta}_K \rangle} \end{pmatrix}}_{\underline{\Phi} = \underline{\Phi}^{(1)*T} \cdot \underline{\Phi}^{(2)}}$$

- idea:
  Once the diagonal elements of $\underline{\Phi}$, i.e., the eigenvalues

$$\underline{\lambda}_k = e^{j\langle \vec{r}^{(2)} - \vec{r}^{(1)}, \vec{\beta}_k \rangle}$$

of $\underline{\Phi}$ are known, it is easy to compute the directions of arrival.

Signal Subspace

- Both the K columns
  - of $\underline{U}_s$ and
  - of $\underline{A}$

  span the same signal subspace.

  $\Rightarrow \underline{U}_s = \underline{A} \cdot \underline{C}'$ with a properly chosen invertible $K \times K$ matrix $\underline{C}'$

- two subarrays:

$$\underline{S}^{(1)} \cdot \underline{U}_s = \underline{S}^{(1)} \cdot \underline{A} \cdot \underline{C}' = \underline{A}^{(1)} \cdot \underline{C}'$$

$$\Rightarrow \underline{A}^{(1)} = \underline{S}^{(1)} \cdot \underline{U}_s \cdot \underline{C}'^{-1}$$

$$\underline{S}^{(2)} \cdot \underline{U}_s = \underline{S}^{(2)} \cdot \underline{A} \cdot \underline{C} = \underline{A}^{(2)} \cdot \underline{C}'$$

$$= \underline{A}^{(1)} \cdot \underline{\Phi} \cdot \underline{C}' = \underline{S}^{(1)} \cdot \underline{U}_s \cdot \underline{C}^{-1} \cdot \underline{\Phi} \cdot \underline{C}'$$

$$\Rightarrow \underline{S}^{(2)} \cdot \underline{U}_s = \underline{S}^{(1)} \cdot \underline{U}_s \cdot \underbrace{\underline{C}'^{-1} \cdot \underline{\Phi} \cdot \underline{C}'}_{\underline{\Psi}}$$

Compute $\underline{\Psi}$ from this linear system of equations.

- $\underline{\Phi}$ and $\underline{\Psi}$ are similar matrices

  $\Rightarrow$ The eigenvalues λ_k of $\underline{\Phi}$ are identical to the eigenvalues of $\underline{\Psi}$.

ESPRIT Algorithm

1.) eigendecomposition of $\underline{R}_{ee}$

2.) the K principal eigenvectors of $\underline{R}_{ee}$ become the columns of $\underline{U}_s$

3.) compute $\underline{\Psi}$ based on $S^{(1)} \cdot \underline{U}_s$ and $S^{(2)} \cdot \underline{U}_s$

4.) the directions of arrival are computed from the eigenvalues λ_k of $\underline{\Psi}$, e.g., in the case of a uniform linear array with two maximally overlapping subarrays:

$$\lambda_k = e^{j d \beta \cos(\varphi_k)}$$

$$\Rightarrow \varphi_k = \arccos\left(\frac{\arg(\lambda_k)}{d \beta}\right)$$

LS-ESPRIT

$$S^{(2)} \cdot \underline{U}_s = S^{(1)} \cdot \underline{U}_s \cdot \underline{\Psi}$$

is an overdetermined linear system of equations for $\underline{\Psi}$ (if there are at least K antenna elements in each subarray).

$\Rightarrow$ least squares pseudosolution

$$\hat{\underline{\Psi}} = \left(S^{(1)} \cdot \underline{U}_s \right)^{+} \cdot S^{(2)} \cdot \underline{U}_s$$

minimizing the error

$$\| S^{(1)} \cdot \underline{U}_s \cdot \underline{\Psi} - S^{(2)} \cdot \underline{U}_s \|_F^2$$

"only $S^{(2)} \cdot \underline{U}_s$ contains errors"

```
function phi = lsesprit(Ree, K, d, varargin)
%LSESPRIT least squares estimation of signal parameters
%   via rotational invariance techniques
%   requires a uniform linear array
%   phi: directions of arrival
%   Ree: correlation matrix
%   K: number of sources
%   d: distance of antenna elements normalized to the wavelength

% eigendecomposition
[U, L] = eig((Ree+Ree')/2, 'vector'); % make sure that the matrix is Hermitian
[L, index] = sort(L, 'descend');
U = U(:,index);
Us = U(:,1:K); % signal subspace

% least squares
psi = pinv(Us(1:end-1,:))*Us(2:end,:);

% directions of arrival
phi = acos(angle(eig(psi))/(2*pi*d));

end
```